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Research Article

A Finite Difference Scheme for the Time-Fractional Landau-Lifshitz-Bloch Equation

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Abstract. In this paper, we study a finite difference scheme for temporal discretization of the time-fractional Landau-Lifshitz-Bloch equation, such as the fractional time derivative of order α is taken in the sense of Caputo. An existence result is established for the semi-discrete problem by Schaefer's fixed point theorem. Stability and error analysis are then provided, showing that the temporal accuracy is of order $2-\alpha$.

Keywords: Landau-Lifshitz-Bloch equation, finite difference method, fractional differential equations, Schaefer's fixed point, existence, stability, error analysis

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1. Introduction

The conventional Landau-Lifshitz (LL) or Landau-Lifshitz-Gilbert (LLG) equation is a grand master equation that provides quantitative predictions on magnetic structure and magnetization dynamics of ferromagnets at low temperatures. At high temperatures, however, the LLG equation fails to describe the longitudinal relaxation and thus it is necessary to extend the LL equation. In particular, it is desirable to obtain a similarly effective equation, which is capable of quantitatively computing magnetization dynamics at high temperatures in real time. Such extension is not only theoretically interesting but also technologically relevant. For high temperatures the LLG equation must be replaced by a more thermodynamically consistent approach such as the Landau-Lifshitz-Bloch (LLB) equation [3]. The LLB equation essentially interpolates between the LLG equation at low temperatures and the Ginzburg-Landau theory of phase transitions. It is valid not only below but also above the Curie temperature T_c. An important property of the LLB equation is that the magnetization magnitude is no longer conserved but is a dynamical variable [4]. The spin polarization vector $\mathbf{u}(x,t)$, where $\mathbf{u} = \frac{\mathbf{m}}{m_s^0}$, and $\mathbf{m}$ is magnetization vector and m_s^0 is the saturation magnetization value at $T=0$. For $\Omega \subset \mathbb{R}^d$, $d \geq 1$, $\mathbf{u}$ satisfies the following LLB equation

$$\frac{\partial \mathbf{u}}{\partial t} = \gamma \mathbf{u} \times H_{\text{eff}}(\mathbf{u}) + \frac{L_1}{|\mathbf{u}|^2}\left(\mathbf{u}\cdot H_{\text{eff}}(\mathbf{u})\right)\mathbf{u} - \frac{L_2}{|\mathbf{u}|^2}\mathbf{u}\times\left(\mathbf{u}\times H_{\text{eff}}(\mathbf{u})\right), \tag{1}$$

where $\gamma > 0$ is the gyromagnetic ratio, the symbol $\times$ denotes the vector cross product in $\mathbb{R}^3$, L_1 and L_2 are the longitudinal and transverse damping parameters, respectively.

Here, we consider a ferromagnetic LLB equation, in which the temperature T is raised higher than T_c, and as a consequence the longitudinal L_1 and transverse L_2 damping parameters are equal. The effective field $H_{\text{eff}}(\mathbf{u})$ is given by

$$H_{\text{eff}}(\mathbf{u}) = \Delta\mathbf{u} - \frac{1}{\chi_{\|}}\left(1 + \frac{3}{5}\frac{T}{T - T_c}|\mathbf{u}|^2\right)\mathbf{u},$$

where $\chi_{\|}$ is the longitudinal susceptibility.

Using the fact that $L_1 = L_2 = \kappa_1$, we can rewrite (1) in the following form

$$\frac{\partial\mathbf{u}}{\partial t} = \kappa_1\Delta\mathbf{u} + \gamma\mathbf{u}\times\Delta\mathbf{u} - \kappa_2(1 + \mu|\mathbf{u}|^2)\mathbf{u}, \quad \text{in } \Omega\times(0,T) \tag{2}$$

where $\kappa_2 = \frac{\kappa_1}{\chi_{\|}}$, and $\mu = \frac{3T}{5(T-T_c)}$.

As boundary and initial conditions we assume

$$\partial_\nu\mathbf{u} = 0, \quad \text{on } \partial\Omega\times(0,T) \tag{3}$$

$$\mathbf{u}(x,0)=\mathbf{u}_0(x) \quad \text{in } \Omega \tag{4}$$

where $\partial_\nu\mathbf{u}$ denotes the outward normal derivative of $\mathbf{u}$ on the boundary of Ω.

For the problem (2)–(4), existence of weak solutions has been proved by Kim Ngan Le (see [7]), using Faedo-Galerkin approximations.

In this paper, we consider the following time-fractional LLB equation in one dimension of space, which is obtained from (2) by replacing the first-order time derivative with a fractional derivative in the Caputo sense:

$$\frac{\partial^\alpha\mathbf{u}(x,t)}{\partial t^\alpha} = \kappa_1\Delta\mathbf{u} + \gamma\mathbf{u}\times\Delta\mathbf{u} - \kappa_2(1 + \mu|\mathbf{u}|^2)\mathbf{u} \quad \text{in } \Omega\times(0,T), \tag{5}$$

where κ_1, κ_2 and μ are positive constants. Equation (5) is subject to the boundary and initial conditions (3)–(4) and $0 < \alpha < 1$, is the order of the time-fractional derivative, $\frac{\partial^\alpha\mathbf{u}(x,t)}{\partial t^\alpha}$ denotes the Caputo fractional derivative of order α as defined in [5] and given by

$$\frac{\partial^\alpha\mathbf{u}(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial\mathbf{u}(x,s)}{\partial s}\frac{ds}{(t-s)^\alpha}, \quad 0 < \alpha < 1.$$

Note that for $\alpha = 1$, we obtain the classical LLB equation (2). Theorem 3 discusses the limit $\alpha \to 1$. When $\alpha = 0$, we obtain the following steady state problem

$$\mathbf{u} = \kappa_1\Delta\mathbf{u} + \gamma\mathbf{u}\times\Delta\mathbf{u} - \kappa_2(1 + \mu|\mathbf{u}|^2)\mathbf{u}$$

with homogeneous Neumann boundary condition. We remark that the unique solution of this problem is $\mathbf{u} = 0$. In recent years, it has been shown that the fractional differential equations can be used successfully to model many phenomena in various fields, such as fluid mechanics, viscoelasticity, chemistry and engineering [1, 6, 10, 11]. Fractional derivative *is* an excellent tool for describing the memory and hereditary properties of various materials and processes while in integer-order models such effects are neglected. It also appears in the theory of control of dynamical systems, where for the description of the controlled system and the controller fractional differential equations are used. Several works on the literature deal with numerical approximation of fractional models. For example in [8], the authors studied the numerical resolution of a time-fractional diffusion equation, which is obtained from the standard diffusion equation by replacing the first-order time derivative with a fractional derivative. Note that the model considered in [8] is linear. In our case the LLB equation is nonlinear and the nonlinearity makes difficult the mathematical analysis of the model equation.

Throughout, we make use of the following notation. For Ω an open bounded domain of $\mathbb{R}$, $k \in \mathbb{N}^*$ *and* $p \geq 1$, we denote by $\mathbb{L}^p(\Omega) = (L^p(\Omega))^3$ and $\mathbb{H}^k(\Omega) = (H^k(\Omega))^3$ the classical Hilbert spaces equipped with the usual norm denoted by $\|\cdot\|_{\mathbb{L}^p(\Omega)}$ and $\|\cdot\|_{\mathbb{H}^k(\Omega)}$.

The rest of the paper is divided as follows. In the next section, a finite difference scheme for the temporal discretization of the problem in consideration is given. We obtain existence of weak solutions to the discretized problem. In Section 3, stability results are derived and error estimates are provided for the semi-discrete problem, showing that the temporal accuracy is of order $2-\alpha$. The last section concludes the paper and provides future directions for this work.

2. A Finite Difference Scheme for Time Fractional LLB

We proceed as in [8, 12, 13]. We introduce a finite difference approximation to discretize the time-fractional derivative. Let $\delta = \frac{T}{N}$ be the length of each time step, for some large N, $t_k = k\delta$, $k = 0, 1, \ldots, N$. We use the following formulation: for all $0 \leq k \leq N-1$;

$$\begin{aligned}\frac{\partial^\alpha \mathbf{u}(x,t)}{\partial t^\alpha} &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} \frac{\partial \mathbf{u}(x,t)}{\partial s} \frac{ds}{(t_{k+1}-s)^\alpha} \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \frac{\mathbf{u}(x,t_{j+1}) - \mathbf{u}(x,t_j)}{\delta} \int_{t_j}^{t_{j+1}} \frac{ds}{(t_{k+1}-s)^\alpha} + \mathbf{r}_\delta^{k+1}\end{aligned} \tag{6}$$

where $\mathbf{r}_\delta^{k+1}$ is the truncation error. It can be seen from [8] that the truncation error verifies

$$\mathbf{r}_\delta^{k+1} \lesssim c_{\mathbf{u}} \delta^{2-\alpha} \tag{7}$$

where $c_{\mathbf{u}}$ is a constant depending only on $\mathbf{u}$. On the other hand, we have

$$\begin{aligned}&\frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \frac{\mathbf{u}(x,t_{j+1}) - \mathbf{u}(x,t_j)}{\delta} \int_{t_j}^{t_{j+1}} \frac{ds}{(t_{k+1}-s)^\alpha} \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \frac{\mathbf{u}(x,t_{j+1}) - \mathbf{u}(x,t_j)}{\delta} \int_{t_{k-j}}^{t_{k+1-j}} \frac{dt}{t^\alpha} \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \frac{\mathbf{u}(x,t_{k+1-j}) - \mathbf{u}(x,t_{k-j})}{\delta} \int_{t_j}^{t_{j+1}} \frac{dt}{t^\alpha} \\ &= \frac{1}{\Gamma(2-\alpha)} \sum_{j=0}^{k} \frac{\mathbf{u}(x,t_{k+1-j}) - \mathbf{u}(x,t_{k-j})}{\delta^\alpha} \left((j+1)^{1-\alpha} - j^{1-\alpha}\right).\end{aligned}$$

Let us denote $b_j = (j+1)^{1-\alpha} - j^{1-\alpha}$, $j = 0, 1, \ldots, k$, and define the discrete fractional differential operator L_t^α by

$$L_t^\alpha \mathbf{u}(x, t_{k+1}) = \frac{1}{\Gamma(2-\alpha)} \sum_{j=0}^{k} b_j \frac{\mathbf{u}(x, t_{k+1-j}) - \mathbf{u}(x, t_{k-j})}{\delta^\alpha}.$$

Then (6) becomes

$$\frac{\partial^\alpha \mathbf{u}(x,t)}{\partial t^\alpha} = L_t^\alpha \mathbf{u}(x, t_{k+1}) + \mathbf{r}_\delta^{k+1}.$$

Using this approximation, we obtain the following finite difference scheme to (2): for $k = 1, \ldots, N-1$,

$$L_t^\alpha \mathbf{u}(x, t_{k+1}) = \kappa_1 \Delta \mathbf{u}^{k+1} + \gamma \mathbf{u}^{k+1} \times \Delta \mathbf{u}^{k+1} - \kappa_2 (1 + \mu |\mathbf{u}^{k+1}|^2) \mathbf{u}^{k+1} \quad \text{in } \Omega \tag{8}$$

where $\mathbf{u}^{k+1}$ is an approximation to $\mathbf{u}(x, t_{k+1})$. The scheme (8) can be reformulated into the form

$$\begin{aligned} b_0 \mathbf{u}^{k+1} - \kappa_1 \Gamma(2-\alpha)\delta^\alpha \Delta \mathbf{u}^{k+1} &= b_0 \mathbf{u}^k - \sum_{j=1}^{k} b_j (\mathbf{u}^{k+1-j} - \mathbf{u}^{k-j}) \\ &+ \gamma \Gamma(2-\alpha)\delta^\alpha \mathbf{u}^{k+1} \times \Delta \mathbf{u}^{k+1} - \kappa_2 \Gamma(2-\alpha)\delta^\alpha (1 + \mu |\mathbf{u}^{k+1}|^2) \mathbf{u}^{k+1} \\ &= b_0 \mathbf{u}^k + \sum_{j=0}^{k-1} (b_j - b_{j+1}) \mathbf{u}^{k-j} + \gamma \Gamma(2-\alpha)\delta^\alpha \mathbf{u}^{k+1} \times \Delta \mathbf{u}^{k+1} \\ &- \kappa_2 \Gamma(2-\alpha)\delta^\alpha (1 + \mu |\mathbf{u}^{k+1}|^2) \mathbf{u}^{k+1}. \end{aligned} \tag{9}$$

To complete the semi-discrete problem, we consider the boundary and initial conditions

$$\partial_\nu \mathbf{u}^{k+1} = 0 \quad \text{on } \partial\Omega,$$

$$\mathbf{u}^0 = \mathbf{u}_0 \quad \text{in } \Omega.$$

Noting that

$$b_j > 0, \quad j = 0, 1, \ldots, k,$$

$$1 = b_0 > b_1 > \ldots > b_k, \quad b_k \to 0 \quad \text{as } k \to \infty,$$

$$\sum_{j=0}^{k} (b_j - b_{j+1}) + b_{k+1} = (1 - b_1) + \sum_{j=1}^{k-1} (b_j - b_{j+1}) + b_k = 1.$$

Letting

$$\beta = \Gamma(2-\alpha)\delta^\alpha$$

then (9) can be rewritten in the form

$$\begin{aligned} &\mathbf{u}^{k+1} - \beta \kappa_1 \Delta \mathbf{u}^{k+1} \\ &= (1 - b_1)\mathbf{u}^k + \sum_{j=1}^{k-1} (b_j - b_{j+1}) \mathbf{u}^{k-j} + b_k \mathbf{u}^0 + \beta \gamma \mathbf{u}^{k+1} \times \Delta \mathbf{u}^{k+1} - \beta \kappa_2 (1 + \mu |\mathbf{u}^{k+1}|^2) \mathbf{u}^{k+1} \end{aligned} \tag{10}$$

for all $k \geq 1$. When $k = 0$, we obtain

$$\mathbf{u}^1 - \beta\kappa_1 \Delta \mathbf{u}^1 = \mathbf{u}^0 + \beta\gamma \mathbf{u}^1 \times \Delta \mathbf{u}^1 - \beta\kappa_2 (1 + \mu |\mathbf{u}^1|^2) \mathbf{u}^1.$$

When $k = 1$, we have

$$\mathbf{u}^2 - \beta\kappa_1 \Delta \mathbf{u}^2 = (1 - b_1)\mathbf{u}^1 + b_1 \mathbf{u}^0 + \beta\gamma \mathbf{u}^2 \times \Delta \mathbf{u}^2 - \beta\kappa_2 (1 + \mu |\mathbf{u}^2|^2) \mathbf{u}^2.$$

We define the error term $\mathbf{r}^{k+1}$ by

$$\mathbf{r}^{k+1} = \beta \Big(\frac{\partial^\alpha \mathbf{u}(x, t_{k+1})}{\partial t^\alpha} - L_t^\alpha \mathbf{u}(x, t_{k+1}) \Big).$$

From (7), it follows that

$$|\mathbf{r}^{k+1}| = \Gamma(2 - \alpha)\delta^\alpha |\mathbf{r}_\delta^{k+1}| \leq c_{\mathbf{u}} \delta^2. \tag{11}$$

2.1. Existence for the Semi-discrete Scheme

What is interesting in this problem is the nonlinear term $\mathbf{u} \times \Delta \mathbf{u}$, this term creates a difficulty concerning the existence of solutions for the discretized problem, that is why we are reduced to study the problem in one dimension to use the Sobolev embedding $\mathbb{H}^1(\Omega) \hookrightarrow \mathbb{L}^\infty(\Omega)$ which implies that $\mathbf{u} \times \nabla \mathbf{u} \in \mathbb{L}^2(\Omega)$ for $\mathbf{u} \in \mathbb{H}^1(\Omega)$.

Before starting, we give the following definition of weak solution of (10).

Definition 1. We say that $\mathbf{u}^{k+1}$ is a weak solution of (10) if

$$\begin{aligned} &\int_\Omega \mathbf{u}^{k+1} \cdot \mathbf{v}\, dx + \beta\kappa_1 \int_\Omega \nabla \mathbf{u}^{k+1} \cdot \nabla \mathbf{v}\, dx \\ &\quad = \int_\Omega \mathbf{f}^k \cdot \mathbf{v}\, dx - \beta\gamma \int_\Omega \mathbf{u}^{k+1} \times \nabla \mathbf{u}^{k+1} \cdot \nabla \mathbf{v}\, dx - \beta\kappa_2 \int_\Omega (1 + \mu |\mathbf{u}^{k+1}|^2) \mathbf{u}^{k+1} \cdot \mathbf{v}\, dx \end{aligned} \tag{12}$$

for all $\mathbf{v} \in \mathbb{H}^1(\Omega)$, where $\mathbf{f}^k = (1 - b_1)\mathbf{u}^k + \sum_{j=1}^{k-1} (b_j - b_{j+1}) \mathbf{u}^{k-j} + b_k \mathbf{u}^0$.

At each time step we have to solve a discretized fractional LLB equation.

Theorem 1. *There exists at least a weak solution* $\mathbf{u}^{k+1}$ *of (9) such that* $\mathbf{u}^{k+1} \in \mathbb{H}^1(\Omega)$.

We proceed with the derivation of a priori estimates. From now on we denote by C a generic constant, which may not be the same at different occurrences.

Lemma 1. *For all k, one has*

$$\|\mathbf{u}^{k+1}\|_{\mathbb{H}^1(\Omega)} \leq C,$$

where C is a positive constant independent of k.

Proof. We prove this result by recurrence. When $k = 0$, we have

$$\begin{aligned} &\int_\Omega \mathbf{u}^1 \cdot \mathbf{v}\, dx + \beta\kappa_1 \int_\Omega \nabla \mathbf{u}^1 \cdot \nabla \mathbf{v}\, dx \\ &\quad = \int_\Omega \mathbf{u}^0 \cdot \mathbf{v}\, dx - \beta\gamma \int_\Omega \mathbf{u}^1 \times \nabla \mathbf{u}^1 \cdot \nabla \mathbf{v}\, dx - \beta\kappa_2 \int_\Omega (1 + \mu |\mathbf{u}^1|^2) \mathbf{u}^1 \cdot \mathbf{v}\, dx \end{aligned}$$

for all $\mathbf{v} \in \mathbb{H}^1(\Omega)$. Taking $\mathbf{v} = \mathbf{u}^1$ in the previous equation, we obtain

$$\int_\Omega |\mathbf{u}^1|^2\,dx + \beta\kappa_1 \int_\Omega |\nabla \mathbf{u}^1|^2\,dx + \beta\kappa_2 \int_\Omega (1+\mu|\mathbf{u}^1|^2)|\mathbf{u}^1|^2\,dx$$
$$= \int_\Omega \mathbf{u}^0 \cdot \mathbf{u}^1\,dx \le \frac{1}{2}\int_\Omega |\mathbf{u}^1|^2\,dx + \frac{1}{2}\int_\Omega |\mathbf{u}^0|^2\,dx.$$

Then

$$\frac{1}{2}\int_\Omega |\mathbf{u}^1|^2\,dx + \beta\kappa_1 \int_\Omega |\nabla \mathbf{u}^1|^2\,dx + \beta\kappa_2 \int_\Omega (1+\mu|\mathbf{u}^1|^2)|\mathbf{u}^1|^2\,dx \le C.$$

Hence

$$\|\mathbf{u}^1\|_{\mathbb{H}^1(\Omega)} \le C.$$

Suppose now that we have

$$\|\mathbf{u}^j\|_{\mathbb{H}^1(\Omega)} \le C; \quad j = 1, 2, \dots, k,$$

and prove that

$$\|\mathbf{u}^{k+1}\|_{\mathbb{H}^1(\Omega)} \le C.$$

Multiplying equation (10) by $\mathbf{u}^{k+1}$ and integrating over Ω, we get

$$\int_\Omega |\mathbf{u}^{k+1}|^2\,dx + \beta\kappa_1 \int_\Omega |\nabla \mathbf{u}^{k+1}|^2\,dx + \beta\kappa_2 \int_\Omega (1+\mu|\mathbf{u}^{k+1}|^2)|\mathbf{u}^{k+1}|^2\,dx$$
$$= \int_\Omega \mathbf{f}^k \cdot \mathbf{u}^{k+1}\,dx \le \frac{1}{2}\int_\Omega |\mathbf{u}^{k+1}|^2\,dx + \frac{1}{2}\int_\Omega |\mathbf{f}^k|^2\,dx.$$

Using the recurrence hypothesis, we have

$$\|\mathbf{f}^k\|_{\mathbb{H}^1(\Omega)} \le C.$$

Then

$$\frac{1}{2}\int_\Omega |\mathbf{u}^{k+1}|^2\,dx + \beta\kappa_1 \int_\Omega |\nabla \mathbf{u}^{k+1}|^2\,dx + \beta\kappa_2 \int_\Omega (1+\mu|\mathbf{u}^{k+1}|^2)|\mathbf{u}^{k+1}|^2\,dx \le C.$$

Therefore

$$\|\mathbf{u}^{k+1}\|_{\mathbb{H}^1(\Omega)} \le C.$$

This concludes the proof of Lemma 1. □

2.2. Proof of Theorem 1

The proof uses Schaefer's fixed point theorem. We construct an appropriate mapping whose fixed points will be solutions to (12). Let $\mathbf{z} \in \mathbb{H}^1(\Omega)$, and define the functional $\mathbf{F}_\mathbf{z}^k \in \mathbb{H}^{-1}(\Omega)$ by

$$\int_\Omega \mathbf{F}_\mathbf{z}^k \cdot \mathbf{v}\,dx = \int_\Omega \mathbf{f}^k \cdot \mathbf{v}\,dx - \beta\gamma \int_\Omega \mathbf{z} \times \nabla\mathbf{z} \cdot \nabla\mathbf{v}\,dx - \beta\kappa_2 \int_\Omega (1+\mu|\mathbf{z}|^2)\mathbf{z}\cdot\mathbf{v}\,dx$$

for all $\mathbf{v} \in \mathbb{H}^1(\Omega)$.

Using Lemma 1, and the fact that $\mathbb{H}^1(\Omega) \hookrightarrow \mathbb{L}^\infty(\Omega)$ in one dimension, we can show that there exists a constant $C > 0$, independent of k, such that

$$\|\mathbf{F}^k_{\mathbf{z}}\|_{\mathbb{H}^{-1}(\Omega)} \leq C.$$

Indeed, we have respectively

$$\Big| \int_\Omega \mathbf{f}^k \cdot \mathbf{v}\, dx \Big| \leq \|\mathbf{f}^k\|_{\mathbb{L}^2(\Omega)} \|\mathbf{v}\|_{\mathbb{L}^2(\Omega)} \leq C\|\mathbf{v}\|_{\mathbb{L}^2(\Omega)} \leq C\|\mathbf{v}\|_{\mathbb{H}^1(\Omega)},$$

$$\Big| \int_\Omega \mathbf{z} \times \nabla \mathbf{z} \cdot \nabla \mathbf{v}\, dx \Big| \leq \|\mathbf{z}\|_\infty \|\nabla \mathbf{z}\|_{\mathbb{L}^2(\Omega)} \|\nabla \mathbf{v}\|_{\mathbb{L}^2(\Omega)} \leq C\|\mathbf{v}\|_{\mathbb{H}^1(\Omega)},$$

and

$$\Big| \int_\Omega (1 + \mu|\mathbf{z}|^2)\mathbf{z} \cdot \mathbf{v}\, dx \Big| \leq (\|\mathbf{z}\|_\infty + \mu\|\mathbf{z}\|^3_\infty) \int_\Omega |\mathbf{v}|\, dx \leq C\|\mathbf{v}\|_{\mathbb{L}^2(\Omega)} \leq C\|\mathbf{v}\|_{\mathbb{H}^1(\Omega)}.$$

Hence, we define the operator $\Psi : \mathbb{H}^1(\Omega) \longmapsto \mathbb{H}^1(\Omega)$ as follows: $\Psi \mathbf{z} = \boldsymbol{\omega}_{\mathbf{z}}$ where $\boldsymbol{\omega}_{\mathbf{z}}$ is the unique solution in $\mathbb{H}^1(\Omega)$ of

$$\int_\Omega \boldsymbol{\omega}_{\mathbf{z}} \cdot \mathbf{v}\, dx + \beta\kappa_1 \int_\Omega \nabla \boldsymbol{\omega}_{\mathbf{z}} \cdot \nabla \mathbf{v}\, dx = \int_\Omega \mathbf{F}^k_{\mathbf{z}} \cdot \mathbf{v}\, dx \tag{13}$$

for all $\mathbf{v} \in \mathbb{H}^1(\Omega)$.

We now assert that Ψ is weakly sequentially continuous and weakly compact. Indeed, let $(\mathbf{z}_n)_n \subset \mathbb{H}^1(\Omega)$, such that $\mathbf{z}_n \rightharpoonup \mathbf{z}$ in $\mathbb{H}^1(\Omega)$. Put $\mathbf{y}_n = \Psi \mathbf{z}_n$, for a subsequence, we have

$$\mathbf{z}_n \to \mathbf{z} \quad \text{strongly in } \mathbb{L}^2(\Omega) \quad \text{and a.e},$$

and there exists $\mathbf{y} \in \mathbb{H}^1(\Omega)$ such that

$$\mathbf{y}_n \rightharpoonup \mathbf{y} \quad \text{weakly in } \mathbb{H}^1(\Omega).$$

We will show that $\mathbf{y}$ is a solution of (13). Indeed, for $\mathbf{v} \in \mathbb{H}^1(\Omega)$, we have

$$\int_\Omega \mathbf{y}_n \cdot \mathbf{v}\, dx + \beta\kappa_1 \int_\Omega \nabla \mathbf{y}_n \cdot \nabla \mathbf{v}\, dx = \int_\Omega \mathbf{F}^k_{\mathbf{z}_n} \cdot \mathbf{v}\, dx. \tag{14}$$

On the other hand, we have

$$\int_\Omega \mathbf{F}^k_{\mathbf{z}_n} \cdot \mathbf{v}\, dx = \int_\Omega \mathbf{f}^k \cdot \mathbf{v}\, dx - \beta\gamma \int_\Omega \mathbf{z}_n \times \nabla \mathbf{z}_n \cdot \nabla \mathbf{v}\, dx - \beta\kappa_2 \int_\Omega (1 + \mu|\mathbf{z}_n|^2)\mathbf{z}_n \cdot \mathbf{v}\, dx.$$

By the previous convergences and continuous embedding $\mathbb{H}^1(\Omega) \hookrightarrow \mathbb{L}^4(\Omega)$, we obtain

$$\int_\Omega \mathbf{F}^k_{\mathbf{z}_n} \cdot \mathbf{v}\, dx \to \int_\Omega \mathbf{F}^k_{\mathbf{z}} \cdot \mathbf{v}\, dx \tag{15}$$

for all $\mathbf{v} \in C^\infty(\overline{\Omega})$, and by density argument (15) holds for all $\mathbf{v} \in \mathbb{H}^1(\Omega)$. In fact, the strong convergence of $\mathbf{z}_n$ to $\mathbf{z}$ in $\mathbb{L}^2(\Omega)$ and the weak convergence of $\nabla \mathbf{z}_n$ to $\nabla \mathbf{z}$ in $\mathbb{L}^2(\Omega)$ allow to deduce that

$$\int_\Omega \mathbf{z}_n \times \nabla \mathbf{z}_n \cdot \mathbf{v}\, dx \to \int_\Omega \mathbf{z} \times \nabla \mathbf{z} \cdot \mathbf{v}\, dx,$$

for all $\mathbf{v} \in C^\infty(\overline{\Omega})$. On the other hand, by the continuous embedding $\mathbb{H}^1(\Omega) \hookrightarrow \mathbb{L}^4(\Omega)$ we obtain that the sequence $(1 + \mu|\mathbf{z}_n|^2)_n$ is bounded in $L^2(\Omega)$. Hence $1 + \mu|\mathbf{z}_n|^2 \rightharpoonup \eta$ in $L^2(\Omega)$. We use the following lemma to show that $\eta = 1 + \mu|\mathbf{z}|^2$.

Lemma 2 (see [9]). *Let Θ be a bounded open subset of $\mathbb{R}_x^d \times \mathbb{R}_t$, h_n and h are functions of $L^q(\Theta)$ with $1 < q < \infty$ such as $\|h_n\|_{L^q(\Theta)} \leq C$, $h_n \to h$ a.e in Θ then $h_n \rightharpoonup h$ weakly in $L^q(\Theta)$.*

Lemma 2 and the strong convergence of $\mathbf{z}_n$ in $\mathbb{L}^2(\Omega)$ allow to deduce that $\eta = 1 + \mu|\mathbf{z}|^2$. Hence

$$(1 + \mu|z_n|^2)\mathbf{z}_n \rightharpoonup (1 + \mu|\mathbf{z}|^2)\mathbf{z} \text{ weakly in } \mathbb{L}^1(\Omega),$$

and then

$$\int_\Omega (1 + \mu|\mathbf{z}_n|^2)\mathbf{z}_n \cdot \mathbf{v}\, dx \to \int_\Omega (1 + \mu|\mathbf{z}|^2)\mathbf{z} \cdot \mathbf{v}\, dx$$

for all $\mathbf{v} \in C^\infty(\overline{\Omega})$.

Now, we can pass to the limit $n \to \infty$ in (14), we get

$$\int_\Omega \mathbf{y} \cdot \mathbf{v}\, dx + \beta\kappa_1 \int_\Omega \nabla\mathbf{y} \cdot \nabla\mathbf{v}\, dx = \int_\Omega \mathbf{F}_\mathbf{z}^k \cdot \mathbf{v}\, dx.$$

Then

$$\Psi\mathbf{z} = \mathbf{y}.$$

Hence Ψ is weakly continuous. A similar argument shows that Ψ is weakly compact, since if $(\mathbf{z}_n)_n$ is bounded in $\mathbb{H}^1(\Omega)$, then $(\Psi\mathbf{z}_n)_n = (\mathbf{y}_n)_n$ is also bounded in $\mathbb{H}^1(\Omega)$, then converge weakly in $\mathbb{H}^1(\Omega)$.

Finally, we must show that the set

$$\{\mathbf{z} \in \mathbb{H}^1(\Omega) / \mathbf{z} = \lambda\Psi\mathbf{z} \text{ for some } 0 < \lambda \leq 1\}$$

is bounded in $\mathbb{H}^1(\Omega)$. So assume $\mathbf{z} \in \mathbb{H}^1(\Omega)$,

$$\mathbf{z} = \lambda\Psi\mathbf{z} \quad \text{for some } 0 < \lambda \leq 1.$$

i.e,

$$\frac{\mathbf{z}}{\lambda} = \Psi\mathbf{z}.$$

Then

$$\int_\Omega \mathbf{z} \cdot \mathbf{v}\, dx + \beta\kappa_1 \int_\Omega \nabla\mathbf{z} \cdot \nabla\mathbf{v}\, dx = \lambda \int_\Omega \mathbf{F}_\mathbf{z}^k \cdot \mathbf{v}\, dx, \tag{16}$$

for all $\mathbf{v} \in \mathbb{H}^1(\Omega)$.

For $\mathbf{v} = \mathbf{z}$, (16) implies that

$$\int_\Omega |\mathbf{z}|^2\, dx + \beta\kappa_1 \int_\Omega |\nabla\mathbf{z}|^2\, dx + \lambda\beta\kappa_2 \int_\Omega (1 + \mu|\mathbf{z}|^2)|\mathbf{z}|^2\, dx$$
$$= \lambda \int_\Omega \mathbf{f}^k \cdot \mathbf{z}\, dx \leq \frac{1}{2}\int_\Omega |\mathbf{z}|^2\, dx + \frac{1}{2}\int_\Omega |\mathbf{f}^k|^2\, dx.$$

Since, by Lemma 1, $(\mathbf{f}^k)_k$ is bounded in $\mathbb{H}^1(\Omega)$ and in particular in $\mathbb{L}^2(\Omega)$, then $\|\mathbf{z}\|_{\mathbb{H}^1(\Omega)} \leq C$, where C is a constant independent of k and λ. Then the proof of Theorem 1 is complete.

Remark 1. We point out that uniqueness can be shown by assuming that

$$\|\nabla \mathbf{u}^{k+1}\|_\infty < \left(\frac{2\kappa_1(\beta\kappa_2+1)}{\beta\gamma^2}\right)^{\frac{1}{2}}.$$

In fact, we can show easily that if $\mathbf{u}^{k+1}$ is a weak solution of the discretized problem then $\mathbf{u}^{k+1} \in \mathbb{H}^2(\Omega)$ which implies that $\nabla \mathbf{u}^{k+1} \in \mathbb{H}^\infty(\Omega)$ in one dimension.

3. Stability and Error Analysis

We have the following unconditionally stability result.

Theorem 2. *The semi-discretized problem (12) is stable, in the sense that for all $\delta > 0$ the following inequality holds*

$$\|\mathbf{u}^{k+1}\|_{\mathbb{H}^1(\Omega)} \leq C\|\mathbf{u}^0\|_{\mathbb{L}^2(\Omega)}.$$

where C is a positive constant independent of k.

Proof. We prove the result by recurrence. When $k = 0$, we have for $\mathbf{v} \in \mathbb{H}^1(\Omega)$ that

$$\int_\Omega \mathbf{u}^1 \cdot \mathbf{v}\, dx + \beta\kappa_1 \int_\Omega \nabla\mathbf{u}^1 \cdot \nabla\mathbf{v}\, dx$$
$$= \int_\Omega \mathbf{u}^0 \cdot \mathbf{v}\, dx - \beta\gamma \int_\Omega \mathbf{u}^1 \times \nabla\mathbf{u}^1 \cdot \nabla\mathbf{v}\, dx - \beta\kappa_2 \int_\Omega (1+\mu|\mathbf{u}^1|^2)\mathbf{u}^1 \cdot \mathbf{v}\, dx.$$

Taking $\mathbf{v} = \mathbf{u}^1$, we obtain

$$\int_\Omega |\mathbf{u}^1|^2\, dx + \beta\kappa_1 \int_\Omega |\nabla\mathbf{u}^1|^2\, dx + \beta\kappa_2 \int_\Omega (1+\mu|\mathbf{u}^1|^2)|\mathbf{u}^1|^2\, dx$$
$$= \int_\Omega \mathbf{u}^0 \cdot \mathbf{u}^1\, dx \leq \|\mathbf{u}^0\|_{\mathbb{L}^2(\Omega)}\|\mathbf{u}^1\|_{\mathbb{L}^2(\Omega)} \leq \|\mathbf{u}^0\|_{\mathbb{L}^2(\Omega)}\|\mathbf{u}^1\|_{\mathbb{H}^1(\Omega)}$$

Then

$$\|\mathbf{u}^1\|_{\mathbb{H}^1(\Omega)} \leq C\|\mathbf{u}^0\|_{\mathbb{L}^2(\Omega)}.$$

Suppose now that we have

$$\|\mathbf{u}^j\|_{\mathbb{H}^1(\Omega)} \leq C\|\mathbf{u}^0\|_{\mathbb{L}^2(\Omega)}, \quad j = 1, 2, \ldots, k,$$

and prove that $\|\mathbf{u}^{k+1}\|_{\mathbb{H}^1(\Omega)} \leq C\|\mathbf{u}^0\|_{\mathbb{L}^2(\Omega)}$. Multiplying equation (10) by $\mathbf{u}^{k+1}$ and integrating over Ω, and using the recurrence hypothesis, we get

$$\begin{aligned}
&\int_\Omega |\mathbf{u}^{k+1}|^2\,dx + \beta\kappa_1 \int_\Omega |\nabla \mathbf{u}^{k+1}|^2\,dx + \beta\kappa_2 \int_\Omega (1+\mu|\mathbf{u}^{k+1}|^2)|\mathbf{u}^{k+1}|^2\,dx \\
&\quad = \int_\Omega \mathbf{f}^k \cdot \mathbf{u}^{k+1}\,dx \\
&\quad \leq (1-b_1)\|\mathbf{u}^k\|_{\mathbb{L}^2(\Omega)}\|\mathbf{u}^{k+1}\|_{\mathbb{L}^2(\Omega)} + \sum_{j=1}^{k-1}(b_j - b_{j+1})\|\mathbf{u}^{k-j}\|_{\mathbb{L}^2(\Omega)}\|\mathbf{u}^{k+1}\|_{\mathbb{L}^2(\Omega)} \\
&\qquad + b_k\|\mathbf{u}^0\|_{\mathbb{L}^2(\Omega)}\|\mathbf{u}^{k+1}\|_{\mathbb{L}^2(\Omega)} \\
&\quad \leq \left((1-b_1) + \sum_{j=1}^{k-1}(b_j - b_{j+1}) + b_k\right)\|\mathbf{u}^0\|_{\mathbb{L}^2(\Omega)}\|\mathbf{u}^{k+1}\|_{\mathbb{L}^2(\Omega)} \\
&\quad \leq \|\mathbf{u}^0\|_{\mathbb{L}^2(\Omega)}\|\mathbf{u}^{k+1}\|_{\mathbb{H}^1(\Omega)}
\end{aligned}$$

since $(1-b_1) + \sum_{j=1}^{k-1}(b_j - b_{j+1}) + b_k = 1$. Then

$$\|\mathbf{u}^{k+1}\|_{\mathbb{H}^1(\Omega)} \leq C\|\mathbf{u}^0\|_{\mathbb{L}^2(\Omega)}.$$

This complete the proof. □

We have the following error analysis for the solution of the semi-discretized problem (12).

Theorem 3. *Let* $\mathbf{u}$ *be the exact solution of (6),* $(\mathbf{u}^j)_j$ *be the time-discrete solution of problem (12) with the initial condition* $\mathbf{u}^0(x) = \mathbf{u}(x,0)$. *Assume further that* $\mathbf{u} \in \mathbb{H}^2(\Omega)$ *such that* $\|\nabla\mathbf{u}\|_\infty < \left(\frac{2\kappa_1(\beta\kappa_2+1)}{\beta\gamma^2}\right)^{\frac{1}{2}}$. *Then we have the following error estimates*

(i) $\|\mathbf{u}(.,t_j) - \mathbf{u}^j\|_{\mathbb{H}^1(\Omega)} \leq \dfrac{C}{1-\alpha}T^\alpha\delta^{2-\alpha}$, $j = 1,\dots,N$ *where* $0 < \alpha < 1$.

(ii) *When* $\alpha \to 1$,

$$\|\mathbf{u}(.,t_j) - \mathbf{u}^j\|_{\mathbb{H}^1(\Omega)} \leq CT\delta, \quad j = 1,\dots,N.$$

Proof. Let $\mathbf{e}^k = \mathbf{u}(x,t_k) - \mathbf{u}^k(x)$ be the difference between the exact solution of (6) and $\mathbf{u}^k$, the time-discrete solution of (12). We will prove the result by induction. We begin with $0 < \alpha < 1$. For $j = 1$, by gathering (6) and (12), the error equation reads

$$\begin{aligned}
&\int_\Omega \mathbf{e}^1 \cdot \mathbf{v}\,dx + \beta\kappa_1 \int_\Omega \nabla\mathbf{e}^1 \cdot \nabla\mathbf{v}\,dx \\
&\quad = \int_\Omega \mathbf{e}^0 \cdot \mathbf{v}\,dx - \int_\Omega \mathbf{r}^1 \cdot \mathbf{v}\,dx + \beta\gamma \int_\Omega (\mathbf{u}^1 \times \nabla\mathbf{u}^1 - \mathbf{u}(x,t_1) \times \nabla\mathbf{u}(x,t_1)) \cdot \nabla\mathbf{v}\,dx \\
&\qquad + \beta\kappa_2 \int_\Omega \left[(1+\mu|\mathbf{u}^1|^2)\mathbf{u}^1 - (1+\mu|\mathbf{u}(x,t_1)|^2)\mathbf{u}(x,t_1)\right] \cdot \mathbf{v}\,dx.
\end{aligned}$$

Choosing $\mathbf{v} = \mathbf{e}^1$ in the above equation, it follows that

$$\int_\Omega |\mathbf{e}^1|^2 \, dx + \beta\kappa_1 \int_\Omega |\nabla \mathbf{e}^1|^2 \, dx$$

$$\leq \|\mathbf{r}^1\|_{\mathbb{L}^2(\Omega)} \|\mathbf{e}^1\|_{\mathbb{L}^2(\Omega)} + \beta\gamma \int_\Omega (\mathbf{u}^1 \times \nabla \mathbf{u}^1 - \mathbf{u}(x,t_1) \times \nabla \mathbf{u}(x,t_1)) \cdot \nabla \mathbf{e}^1 \, dx$$

$$+\beta\kappa_2 \int_\Omega \left[(1+\mu|\mathbf{u}^1|^2)\mathbf{u}^1 - (1+\mu|\mathbf{u}(x,t_1)|^2)\mathbf{u}(x,t_1)\right] \cdot \mathbf{e}^1 \, dx.$$

We have

$$\int_\Omega \left(\mathbf{u}^1 \times \nabla \mathbf{u}^1 - \mathbf{u}(x,t_1) \times \nabla \mathbf{u}(x,t_1)\right) \cdot \nabla \mathbf{e}^1 \, dx$$

$$= \int_\Omega \left(\mathbf{u}^1 \times \nabla \mathbf{u}^1 - \mathbf{u}^1 \times \nabla \mathbf{u}(x,t_1) + \mathbf{u}^1 \times \nabla \mathbf{u}(x,t_1) - \mathbf{u}(x,t_1) \times \nabla \mathbf{u}(x,t_1)\right) \cdot \nabla \mathbf{e}^1 \, dx$$

$$= \int_\Omega \left(\mathbf{u}^1 \times (\nabla \mathbf{u}^1 - \nabla \mathbf{u}(x,t_1)) + (\mathbf{u}^1 - \mathbf{u}(x,t_1)) \times \nabla \mathbf{u}(x,t_1)\right) \cdot \nabla \mathbf{e}^1 \, dx$$

$$= -\int_\Omega \left(\mathbf{u}^1 \times \nabla \mathbf{e}^1 + \mathbf{e}^1 \times \nabla \mathbf{u}(x,t_1)\right) \cdot \nabla \mathbf{e}^1 \, dx$$

$$= -\int_\Omega \mathbf{e}^1 \times \nabla \mathbf{u}(x,t_1) \cdot \nabla \mathbf{e}^1 \, dx.$$

In the same way, we have

$$\int_\Omega \left[(1+\mu|\mathbf{u}^1|^2)\mathbf{u}^1 - (1+\mu|\mathbf{u}(x,t_1)|^2)\mathbf{u}(x,t_1)\right] \cdot \mathbf{e}^1 \, dx$$

$$= \int_\Omega \Big[(1+\mu|\mathbf{u}^1|^2)\mathbf{u}^1 - (1+\mu|\mathbf{u}^1|^2)\mathbf{u}(x,t_1) + (1+\mu|\mathbf{u}^1|^2)\mathbf{u}(x,t_1) - (1+\mu|\mathbf{u}(x,t_1)|^2)\mathbf{u}(x,t_1)\Big]$$

$$\cdot \mathbf{e}^1 \, dx$$

$$= \int_\Omega (1+\mu|\mathbf{u}^1|^2)(\mathbf{u}^1 - \mathbf{u}(x,t_1)) \cdot \mathbf{e}^1 \, dx + \int_\Omega \mu(|\mathbf{u}^1|^2 - |\mathbf{u}(x,t_1)|^2)\mathbf{u}(x,t_1) \cdot \mathbf{e}^1 \, dx$$

$$= -\int_\Omega (1+\mu|\mathbf{u}^1|^2)|\mathbf{e}^1|^2 \, dx + \mu \int_\Omega (\mathbf{u}^1 - \mathbf{u}(x,t_1))(\mathbf{u}^1 + \mathbf{u}(x,t_1))(\mathbf{u}(x,t_1) \cdot \mathbf{e}^1) \, dx$$

$$= -\int_\Omega (1+\mu|\mathbf{u}^1|^2)|\mathbf{e}^1|^2 dx - \mu \int_\Omega ((\mathbf{u}^1 + \mathbf{u}(x,t_1)) \cdot \mathbf{e}^1)(\mathbf{u}(x,t_1) \cdot \mathbf{e}^1) \, dx$$

$$= -\int_\Omega (1+\mu|\mathbf{u}^1|^2)|\mathbf{e}^1|^2 \, dx - \mu \int_\Omega (\mathbf{u}(x,t_1) \cdot \mathbf{e}^1)^2 \, dx - \mu \int_\Omega (\mathbf{u}^1 \cdot \mathbf{e}^1) \cdot (\mathbf{u}(x,t_1) \cdot \mathbf{e}^1) \, dx.$$

The above calculations and Young's inequality allow to get

$$(1+\beta\kappa_2) \int_\Omega |\mathbf{e}^1|^2 \, dx + \beta\kappa_1 \int_\Omega |\nabla \mathbf{e}^1|^2 \, dx + \beta\kappa_2\mu \int_\Omega (\mathbf{u}(x,t_1) \cdot \mathbf{e}^1)^2 \, dx$$

$$+\beta\kappa_2\mu \int_\Omega |\mathbf{u}^1|^2 |\mathbf{e}^1|^2 \, dx \leq \|\mathbf{r}^1\|_{\mathbb{L}^2(\Omega)} \|\mathbf{e}^1\|_{\mathbb{L}^2(\Omega)} + \beta\gamma \int_\Omega |\mathbf{e}^1||\nabla \mathbf{e}^1||\nabla \mathbf{u}(.,t_1)| \, dx \tag{17}$$

$$+\beta\kappa_2\mu \int_\Omega |\mathbf{u}^1|^2 |\mathbf{e}^1|^2 \, dx + \beta\kappa_2\mu \int_\Omega (\mathbf{u}(x,t_1) \cdot \mathbf{e}^1)^2 \, dx.$$

That is

$$(1+\beta\kappa_2)\int_\Omega |\mathbf{e}^1|^2\,dx+\beta\kappa_1\int_\Omega |\nabla\mathbf{e}^1|^2\,dx$$
$$\leq \|\mathbf{r}^1\|_{\mathbb{L}^2(\Omega)}\|\mathbf{e}^1\|_{\mathbb{L}^2(\Omega)}+\beta\gamma\|\nabla\mathbf{u}(.,t_1)\|_\infty\Big(\frac{\varepsilon}{2}\int_\Omega |\mathbf{e}^1|^2\,dx+\frac{1}{2\varepsilon}\int_\Omega |\nabla\mathbf{e}^1|^2\,dx\Big)$$

for $\varepsilon>0$. Hence

$$\Big(1+\beta\kappa_2-\beta\gamma\|\nabla\mathbf{u}(.,t_1)\|_\infty\frac{\varepsilon}{2}\Big)\int_\Omega |\mathbf{e}^1|^2\,dx+\beta\Big(\kappa_1-\gamma\frac{\|\nabla\mathbf{u}(.,t_1)\|_\infty}{2\varepsilon}\Big)\int_\Omega |\nabla\mathbf{e}^1|^2\,dx \leq \|\mathbf{r}^1\|_{\mathbb{L}^2(\Omega)}\|\mathbf{e}^1\|_{\mathbb{H}^1(\Omega)} \quad (18)$$

We choose ε such that

$$1+\beta\kappa_2>\beta\gamma\|\nabla\mathbf{u}(.,t_1)\|_\infty\frac{\varepsilon}{2}\qquad \text{and}\qquad \beta\kappa_1>\beta\gamma\frac{\|\nabla\mathbf{u}(.,t_1)\|_\infty}{2\varepsilon}.$$

This choice is possible since $\|\nabla\mathbf{u}\|_\infty<\left(\frac{2\kappa_1(\beta\kappa_2+1)}{\beta\gamma^2}\right)^{\frac{1}{2}}$.

Dividing both sides by $\|\mathbf{e}^1\|_{\mathbb{H}^1(\Omega)}$, and using (11) we obtain

$$\|\mathbf{u}(.,t_1)-\mathbf{u}^1\|_{\mathbb{H}^1(\Omega)}\leq C\delta^2.$$

for $\varepsilon>0$. Since $1\leq\frac{1}{1-\alpha}$, $b_0=1$ and $\delta\leq T$, we get

$$\|\mathbf{u}(.,t_1)-\mathbf{u}^1\|_{\mathbb{H}^1(\Omega)}\leq \frac{C}{1-\alpha}T^\alpha\delta^{2-\alpha}.$$

Then point (i) is verified for $j=1$. Suppose now we have proved (i) for all $j=1,\dots,k$, and prove it also for $j=k+1$. Combining (5) and (12), we obtain

$$\int_\Omega \mathbf{e}^{k+1}\cdot\mathbf{v}\,dx+\beta\kappa_1\int_\Omega\nabla\mathbf{e}^{k+1}\cdot\nabla\mathbf{v}\,dx$$
$$=(1-b_1)\int_\Omega\mathbf{e}^k\cdot\mathbf{v}\,dx+\sum_{j=1}^{k-1}(b_j-b_{j+1})\int_\Omega\mathbf{e}^{k-j}\cdot\mathbf{v}\,dx+b_k\int_\Omega\mathbf{e}^0\cdot\mathbf{v}\,dx-\int_\Omega\mathbf{r}^{k+1}\cdot\mathbf{v}\,dx$$
$$+\beta\gamma\int_\Omega(\mathbf{u}^{k+1}\times\nabla\mathbf{u}^{k+1}-\mathbf{u}(x,t_{k+1})\times\nabla\mathbf{u}(x,t_{k+1}))\cdot\nabla\mathbf{v}\,dx$$
$$+\beta\kappa_2\int_\Omega\left[(1+\mu|\mathbf{u}^{k+1}|^2)\mathbf{u}^{k+1}-(1+\mu|\mathbf{u}(x,t_{k+1})|^2)\mathbf{u}(x,t_{k+1})\right]\cdot\mathbf{v}\,dx. \quad (19)$$

Taking $\mathbf{v} = \mathbf{e}^{k+1}$ in (19) and by similar calculations used to obtain (17), we get

$$\begin{aligned}
&(1+\beta\kappa_2)\int_\Omega |\mathbf{e}^{k+1}|^2\,dx + \beta\kappa_1\int_\Omega |\nabla\mathbf{e}^{k+1}|^2\,dx + \beta\kappa_2\mu\int_\Omega |\mathbf{u}^{k+1}|^2|\mathbf{e}^{k+1}|^2\,dx\\
&\quad+\beta\kappa_2\mu\int_\Omega (\mathbf{u}(x,t_{k+1})\cdot\mathbf{e}^{k+1})^2\,dx\\
&\le (1-b_1)\|\mathbf{e}^k\|_{\mathbb{L}^2(\Omega)}\|\mathbf{e}^{k+1}\|_{\mathbb{L}^2(\Omega)}\\
&\quad+\sum_{j=1}^{k-1}(b_j-b_{j+1})\|\mathbf{e}^{k-j}\|_{\mathbb{L}^2(\Omega)}\|\mathbf{e}^{k+1}\|_{\mathbb{L}^2(\Omega)} + \|\mathbf{r}^{k+1}\|_{\mathbb{L}^2(\Omega)}\|\mathbf{e}^{k+1}\|_{\mathbb{L}^2(\Omega)}\\
&\quad+\beta\gamma\int_\Omega |\mathbf{e}^{k+1}||\nabla\mathbf{e}^{k+1}||\nabla\mathbf{u}(.,t_{k+1})|\,dx + \beta\kappa_2\mu\int_\Omega |\mathbf{u}^{k+1}|^2|\mathbf{e}^{k+1}|^2\,dx\\
&\quad+\beta\kappa_2\mu\int_\Omega (\mathbf{u}(x,t_{k+1})\cdot\mathbf{e}^{k+1})^2\,dx.
\end{aligned}\tag{20}$$

That is

$$\begin{aligned}
&(1+\beta\kappa_2)\int_\Omega |\mathbf{e}^{k+1}|^2\,dx + \beta\kappa_1\int_\Omega |\nabla\mathbf{e}^{k+1}|^2\,dx\\
&\quad\le (1-b_1)\|\mathbf{e}^k\|_{\mathbb{L}^2(\Omega)}\|\mathbf{e}^{k+1}\|_{\mathbb{L}^2(\Omega)} + \sum_{j=1}^{k-1}(b_j-b_{j+1})\|\mathbf{e}^{k-j}\|_{\mathbb{L}^2(\Omega)}\|\mathbf{e}^{k+1}\|_{\mathbb{L}^2(\Omega)}\\
&\qquad+\|\mathbf{r}^{k+1}\|_{\mathbb{L}^2(\Omega)}\|\mathbf{e}^{k+1}\|_{\mathbb{L}^2(\Omega)} + \beta\gamma\|\nabla\mathbf{u}(.,t_{k+1})\|_\infty\Big(\frac{\varepsilon}{2}\int_\Omega |\mathbf{e}^{k+1}|^2\,dx + \frac{1}{2\varepsilon}\int_\Omega |\nabla\mathbf{e}^{k+1}|^2\,dx\Big)
\end{aligned}$$

for $\varepsilon > 0$. By the same reasoning used for the case $j = 1$ and the induction assumption and the fact that $\frac{b_k^{-1}}{b_{k+1}^{-1}} < 1$, we get

$$\begin{aligned}
&\Big(1+\beta\kappa_2-\beta\gamma\|\nabla\mathbf{u}(.,t_{k+1})\|_\infty\frac{\varepsilon}{2}\Big)\int_\Omega |\mathbf{e}^{k+1}|^2\,dx + \beta\Big(\kappa_1-\gamma\frac{\|\nabla\mathbf{u}(.,t_{k+1})\|_\infty}{2\varepsilon}\Big)\int_\Omega |\nabla\mathbf{e}^{k+1}|^2\,dx\\
&\quad\le\Big((1-b_1)b_{k-1}^{-1} + \sum_{j=1}^{k-1}(b_j-b_{j+1})b_{k-j-1}^{-1}\Big)C\delta^2\|\mathbf{e}^{k+1}\|_{\mathbb{L}^2(\Omega)} + \|\mathbf{r}^{k+1}\|_{\mathbb{L}^2(\Omega)}\|\mathbf{e}^{k+1}\|_{\mathbb{L}^2(\Omega)}\\
&\quad\le\Big((1-b_1) + \sum_{j=1}^{k-1}(b_j-b_{j+1}) + b_k\Big)Cb_{k-1}^{-1}\delta^2\|\mathbf{e}^{k+1}\|_{\mathbb{H}^1(\Omega)}.
\end{aligned}\tag{21}$$

Recall that

$$(1-b_1) + \sum_{j=1}^{k-1}(b_j-b_{j+1}) + b_k = 1.$$

For a suitable choice of ε and dividing both sides by $\|\mathbf{e}^{k+1}\|_{\mathbb{H}^1(\Omega)}$, we obtain

$$\|\mathbf{e}^{k+1}\|_{\mathbb{H}^1(\Omega)} \le Cb_{k-1}^{-1}\delta^2.$$

Noting that

$$k^{-\alpha}b_{k-1}^{-1} \le \frac{1}{1-\alpha},\quad k = 1,\dots,N.$$

Then, we have for all k such that $k\delta \leq T$

$$\begin{aligned} \|\mathbf{u}(.,t_{k+1}) - \mathbf{u}^{k+1}\|_{\mathbb{H}^1(\Omega)} &= \|\mathbf{e}^{k+1}\|_{\mathbb{H}^1(\Omega)} \\ &\leq Ck^{-\alpha} b_{k-1}^{-1} k^{\alpha}\delta^2 \\ &\leq \tfrac{C}{1-\alpha}(k\delta)^{\alpha}\delta^{2-\alpha} \\ &\leq \tfrac{C}{1-\alpha}T^{\alpha}\delta^{2-\alpha}. \end{aligned}$$

Then (i) is proved.

Now, to prove (ii), we will derive again the following estimation by induction:

$$\|\mathbf{u}(.,t_j) - \mathbf{u}^j\|_{\mathbb{H}^1(\Omega)} \leq Cj\delta^2, \quad j = 1, \dots, k. \tag{22}$$

The above inequality is obvious for $j = 1$. Suppose now that (22) holds for all $j = 1, \dots, k$ and prove that it holds also for $j = k+1$. Similarly to the previous case, by combining (5) and (12) and taking $\mathbf{v} = \mathbf{e}^{k+1}$ as a test function and using the induction assumption, we obtain

$$\begin{aligned} (1+\beta\kappa_2)\|\mathbf{e}^{k+1}\|^2_{\mathbb{L}^2(\Omega)} + \beta\kappa_1\|\nabla\mathbf{e}^{k+1}\|^2_{\mathbb{L}^2(\Omega)} &\leq (1-b_1)\|\mathbf{e}^k\|_{\mathbb{L}^2(\Omega)}\|\mathbf{e}^{k+1}\|_{\mathbb{L}^2(\Omega)} \\ &+ \sum_{j=1}^{k-1}(b_j - b_{j+1})\|\mathbf{e}^{k-j}\|_{\mathbb{L}^2(\Omega)}\|\mathbf{e}^{k+1}\|_{\mathbb{L}^2(\Omega)} + \|\mathbf{r}^{k+1}\|_{\mathbb{L}^2(\Omega)}\|\mathbf{e}^{k+1}\|_{\mathbb{L}^2(\Omega)} \\ &+\beta\gamma\|\nabla\mathbf{u}(.,t_{k+1})\|_{\infty}\Big(\frac{\varepsilon}{2}\int_{\Omega}|\mathbf{e}^{k+1}|^2\,dx + \frac{1}{2\varepsilon}\int_{\Omega}|\nabla\mathbf{e}^{k+1}|^2\,dx\Big) \\ &\leq \Big((1-b_1)Ck\delta^2 + \sum_{j=1}^{k-1}(b_j - b_{j+1})C(k-j)\delta^2 + C\delta^2\Big)\|\mathbf{e}^{k+1}\|_{\mathbb{L}^2(\Omega)} \\ &+\beta\gamma\|\nabla\mathbf{u}(.,t_{k+1})\|_{\infty}\Big(\frac{\varepsilon}{2}\int_{\Omega}|\mathbf{e}^{k+1}|^2\,dx + \frac{1}{2\varepsilon}\int_{\Omega}|\nabla\mathbf{e}^{k+1}|^2\,dx\Big) \\ &\leq \Big((1-b_1)\frac{k}{k+1} + \sum_{j=1}^{k-1}(b_j - b_{j+1})\frac{k-j}{k+1} + \frac{1}{k+1}\Big)C(k+1)\delta^2\|\mathbf{e}^{k+1}\|_{\mathbb{L}^2(\Omega)} \\ &+\beta\gamma\|\nabla\mathbf{u}(.,t_{k+1})\|_{\infty}\Big(\frac{\varepsilon}{2}\int_{\Omega}|\mathbf{e}^{k+1}|^2\,dx + \frac{1}{2\varepsilon}\int_{\Omega}|\nabla\mathbf{e}^{k+1}|^2\,dx\Big) \\ &\leq \Big((1-b_1) + \sum_{j=1}^{k-1}(b_j - b_{j+1}) - (1-b_1)\frac{1}{k+1} \\ &- \sum_{j=1}^{k-1}(b_j - b_{j+1})\frac{j+1}{k+1} + \frac{1}{k+1}\Big)C(k+1)\delta^2\|\mathbf{e}^{k+1}\|_{\mathbb{L}^2(\Omega)} \\ &+\beta\gamma\|\nabla\mathbf{u}(.,t_{k+1})\|_{\infty}\Big(\frac{\varepsilon}{2}\int_{\Omega}|\mathbf{e}^{k+1}|^2\,dx + \frac{1}{2\varepsilon}\int_{\Omega}|\nabla\mathbf{e}^{k+1}|^2\,dx\Big). \end{aligned}$$

Since

$$\frac{1-b_1}{k+1} + \sum_{j=1}^{k-1}(b_j - b_{j+1})\frac{j+1}{k+1} + b_k \geq \frac{1}{k+1}\Big((1-b_1) + \sum_{j=1}^{k-1}(b_j - b_{j+1}) + b_k\Big) = \frac{1}{k+1},$$

Then

$$\left(1+\beta\kappa_2-\beta\gamma\|\nabla\mathbf{u}(.,t_{k+1})\|_\infty\frac{\varepsilon}{2}\right)\|\mathbf{e}^{k+1}\|^2_{\mathbb{L}^2(\Omega)}+\beta\left(\kappa_1-\gamma\frac{\|\nabla\mathbf{u}(.,t_{k+1})\|_\infty}{2\varepsilon}\right)\|\nabla\mathbf{e}^{k+1}\|^2_{\mathbb{L}^2(\Omega)}$$
$$\leq\left((1-b_1)+\sum_{j=1}^{k-1}(b_j-b_{j+1})+b_k\right)C(k+1)\delta^2\|\mathbf{e}^{k+1}\|_{\mathbb{H}^1(\Omega)}$$
$$=C(k+1)\delta^2\|\mathbf{e}^{k+1}\|_{\mathbb{H}^1(\Omega)},$$

and it follows, for an ε well chosen and after dividing both sides by $\|\mathbf{e}^{k+1}\|_{\mathbb{H}^1(\Omega)}$ that

$$\|\mathbf{u}(.,t_{k+1})-\mathbf{u}^{k+1}\|_{\mathbb{H}^1(\Omega)}=\|\mathbf{e}^{k+1}\|_{\mathbb{H}^1(\Omega)}\leq C(k+1)\delta^2,$$

for k satisfying $(k+1)\delta\leq T$. We conclude that

$$\|\mathbf{u}(.,t_{k+1})-\mathbf{u}^{k+1}\|_{\mathbb{H}^1(\Omega)}\leq CT\delta.$$

Then *(ii)* is proved, and the proof of Theorem 3 is complete. □

4. Concluding Remarks

In this paper, we proposed a finite difference scheme for the time fractional Landau-Lifshitz-Bloch equation such as the fractional time derivative of order $0<\alpha<1$ is taken in the sense of Caputo. The main challenge here is due to the nonlinearity on the right-hand side of the partial differential equation. A fixed point procedure is used to show the existence of solution for the discretized model. The stability analysis is provided showing that the temporal accuracy is of order $2-\alpha$. We intend to complete the results obtained in this paper by performing a full discretization and conducting numerical experiments for the considered model. These two issues represent an interesting direction of future research.

Competing Interests

The authors declare that they have no competing interests.

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