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Research Article

Existence of Positive Solutions to a Singular Semipositone Boundary Value Problem of Nonlinear Fractional Differential Systems

Xiaofeng Zhang[1] and Hanying Feng[1,2]

[1] *Department of Mathematics, Shijiazhuang Mechanical Engineering College Shijiazhuang 050003, Hebei, P. R. China*
[2] *Department of Mathematics, Nantong Institute of Technology, Nantong 226002, Jiangsu, P. R. China*

Abstract. In this paper, we consider the existence of positive solutions to a singular semipositone boundary value problem of nonlinear fractional differential equations. By applying the fixed point index theorem, some new results for the existence of positive solutions are obtained. In addition, an example is presented to demonstrate the application of our main results.

Keywords: Fractional differential equation, Nonlinear singular semipositone system, Positive solution, Fixed point index

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Corresponding Author
Hanying Feng
fhanying@126.com

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1. Introduction

In this paper, we discuss the following singular semipositone system of nonlinear fractional differential equations:

$$\begin{aligned} &D_{0+}^{\alpha} u(t) + f(t, u(t), v(t)) = 0, \quad 0 < t < 1, \\ &D_{0+}^{\alpha} v(t) + g(t, u(t), v(t)) = 0, \quad 0 < t < 1, \\ &u(0) = u(1) = u'(0) = u'(1) = v(0) = v(1) = v'(0) = v'(1) = 0, \end{aligned} \tag{1}$$

where $3 < \alpha \le 4$ is a real number, D_{0+}^{α} is the standard Riemann-Liouville fractional derivative, and $f, g : (0, 1) \times [0, +\infty) \times [0, +\infty) \to (-\infty, +\infty)$ are given continuous functions. f, g may be singular at $t = 0$ and/or $t = 1$ and may take negative values. By using the fixed point index theorem, some new results for the existence of positive solutions are established.

Singular boundary value problems arise from many fields in physics, biology, chemistry and economics, and play a very important role in both theoretical development and application. Recently, some work has been done to study the existence and multiplicity of solutions or positive solutions of nonlinear singular semipositone boundary value problems by the use of techniques of nonlinear analysis such as Leray-Schauder theory, Krasnoselskii's fixed point theorem, etc [1, 3, 4, 7, 9, 11, 12].

In [7], by using the fixed point index theorem, Liu, Zhang and Wu have studied the existence of positive solutions for a nonlinear singular semipositone system:

$$-x''(t) = f(t, y(t), x(t)) + p(t), \quad t \in (0, 1),$$

$$-y''(t) = g(t, x(t), y(t)) + q(t), \quad t \in (0, 1),$$

$$x(0) = x(1) = 0,$$

$$y(0) = y(1) = 0,$$

where $f, g : (0, 1)\times[0, +\infty)\times[0, +\infty) \to [0, +\infty)$ are continuous and may be singular at $t = 0$ and/or $t = 1$, p and $q : (0, 1) \to (-\infty, +\infty)$ are Lebesgue integrable and may have finitely many singularities in $[0, 1]$.

In [12], Zhu, Liu and Wu have discussed the existence of positive solutions for the fourth-order singular semipositone system:

$$-x^{(4)}(t) = f(t, x(t), y(t), x''(t), y''(t)), \quad t \in (0, 1),$$

$$-y^{(4)}(t) = g(t, x(t), y(t), x''(t), y''(t)), \quad t \in (0, 1),$$

$$x(0) = x(1) = x''(t) = y''(t) = 0,$$

$$y(0) = y(1) = x''(t) = y''(t) = 0,$$

where $f, g : (0, 1)\times[0, +\infty)\times[0, +\infty)\times(-\infty, 0]\times(-\infty, 0] \to (-\infty, +\infty)$ are given continuous functions. f, g may be singular at $t = 0$ and/or $t = 1$ and may take negative values.

In [6], Henderson and Luca have considered the existence of positive solutions for the system of nonlinear fractional differential equations:

$$D^{\alpha}_{0+}u(t) + \lambda f(t, u(t), v(t)) = 0, \quad t \in (0, 1),$$

$$D^{\beta}_{0+}v(t) + \mu g(t, u(t), v(t)) = 0, \quad t \in (0, 1),$$

with the coupled integral boundary conditions

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u'(1) = \int_0^1 v(s)\,\mathrm{d}H(s),$$

$$v(0) = v'(0) = \cdots = v^{(n-2)}(0) = 0, \qquad v'(1) = \int_0^1 u(s)\,\mathrm{d}K(s),$$

where $\alpha \in (n - 1, n], \beta \in (m - 1, m], n, m \in \mathbb{N}, n, m \geq 3$, $D^{\alpha}_{0+}, D^{\beta}_{0+}$ denote the standard Riemann-Liouville fractional derivatives, f, g are sign-changing continuous functions and may be nonsingular or singular at $t = 0$ and/or $t = 1$.

Motivated by the above work, we consider the existence of positive solutions for the system of the fractional order singular semipositone boundary value problem (1).

This paper is organized as follows. In Section 2, we present some basic definitions and properties from the fractional calculus theory. In Section 3, based on the fixed point index theorem, we prove existence theorem of the positive solutions for boundary value problem (1). In section 4, an example is presented to illustrate the main results.

2. Preliminaries

In this section, we present here the necessary definitions and properties from fractional calculus theory. These definitions and properties can be found in the recent literature [2, 5, 8, 10, 11, 13].

Definition 2.1 (see [2]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (0, +\infty) \to \mathbb{R}$ is given by

$$I^{\alpha}_{0+} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,\mathrm{d}s, \quad t > 0,$$

provided the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 2.2 (see [2, 8]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ for a function $y : (0, +\infty) \to \mathbb{R}$ is given by

$$D^{\alpha}_{0+} y(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n (I^{n-\alpha}_{0+} y)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}}\,\mathrm{d}s, \quad t > 0,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α, provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Lemma 2.1 (see [13]). *Let $\alpha > 0$. If we assume $u \in C(0,1) \cap L(0,1)$, then the fractional differential equation*

$$D^{\alpha}_{0+} u(t) = 0$$

has solutions $u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n}, C_i \in \mathbb{R}, i = 1, 2, \ldots, n,\ n = [\alpha] + 1$.

Lemma 2.2 (see [13]). *Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha (\alpha > 0)$ that belongs to $C(0,1) \cap L(0,1)$, then*

$$I^{\alpha}_{0+} D^{\alpha}_{0+} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n},$$

for some $C_i \in \mathbb{R}, i = 1, 2, \ldots, n,\ n = [\alpha] + 1$.

In the following, we present Green's function of the fractional differential equation boundary value problem.

Lemma 2.3 (see [10]). *Let $y \in C[0,1]$ and $3 < \alpha \leq 4$, the unique solution of problem*

$$\begin{gathered} D^{\alpha}_{0+} u(t) + y(t) = 0, \quad 0 < t < 1, \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{gathered} \tag{2}$$

is

$$u(t) = \int_0^1 G(t,s)y(s)\,\mathrm{d}s,$$

where

$$G(t,s) = \begin{cases} \dfrac{(t-s)^{\alpha-1} + (1-s)^{\alpha-2}t^{\alpha-2}\left[(s-t)+(\alpha-2)(1-t)s\right]}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \dfrac{t^{\alpha-2}(1-s)^{\alpha-2}\left[(s-t)+(\alpha-2)(1-t)s\right]}{\Gamma(\alpha)}, & 0 \le t \le s \le 1. \end{cases} \tag{3}$$

Here $G(t,s)$ is called the Green's function of boundary value problem (2).

Lemma 2.4 (see [10, 11]). *The function $G(t,s)$ defined by* (4) *possesses the following properties:*

(1) $G(t,s) > 0$, *for* $t, s \in (0,1)$;

(2) $G(t,s) = G(1-s, 1-t)$, *for* $t, s \in (0,1)$;

(3) $t^{\alpha-2}(1-t)^2 q(s) \le G(t,s) \le (\alpha-1)q(s)$, *for* $t, s \in (0,1)$;

(4) $t^{\alpha-2}(1-t)^2 q(s) \le G(t,s) \le ((\alpha-1)(\alpha-2)/\Gamma(\alpha))t^{\alpha-2}(1-t)^2$, *for* $t, s \in (0,1)$,

where $q(s) = ((\alpha-2)/\Gamma(\alpha))s^2(1-s)^{\alpha-2}$.

Lemma 2.5. *The function $q(1-t)$ has the property:*

$$\max_{t\in(0,1)} q(1-t) = q\left(\frac{2}{\alpha}\right) = \frac{4(\alpha-2)^{\alpha-1}}{\Gamma(\alpha)\alpha^\alpha}.$$

Proof. From the Lemma 2.4, we can easy get $q(1-t) = \frac{\alpha-2}{\Gamma(\alpha)}t^{\alpha-2}(1-t)^2$. Let $F(t) = t^{\alpha-2}(1-t)^2$, since $F'(t) = (1-t)t^{\alpha-3}\left[-\alpha t + (\alpha-2)\right]$, for $t \in (0,1)$, let $F'(t) = 0$, we get $t_0 = \frac{\alpha-2}{\alpha}$.

Since $3 < \alpha \le 4$, we can know $0 < t_0 < 1$. So, the function $F(t)$ achieve the maximum when $t = \frac{\alpha-2}{\alpha}$.

Therefore

$$\max_{t\in(0,1)} F(t) = F\left(\frac{\alpha-2}{\alpha}\right) = \frac{4(\alpha-2)^{\alpha-2}}{\alpha^\alpha},$$

thus,

$$\max_{t\in(0,1)} q(1-t) = q\left(\frac{2}{\alpha}\right) = \frac{4(\alpha-2)^{\alpha-1}}{\Gamma(\alpha)\alpha^\alpha}.$$

For convenience, throughout the rest of the paper, we make the following assumptions:

(H_1) $f, g \in C((0,1)\times[0,+\infty)\times[0,+\infty), (-\infty,+\infty))$ and there exist functions $p_i, a_i, k \in L^1((0,1),[0,+\infty)) \cap C((0,1),[0,+\infty))$ and $h \in C([0,+\infty)\times[0,+\infty),[0,+\infty))$ such that

$$a_1(t)h(x,y) \le f(t,x,y) + p_1(t) \le k(t)h(x,y),$$

$$a_2(t)h(x,y) \le g(t,x,y) + p_2(t) \le k(t)h(x,y),$$

where $a_i(t) \geq c_i k(t)$ a.e. $t \in (0,1)$, $0 < c_i \leq 1$, $i = 1, 2$, $\forall (t, x, y) \in (0,1) \times [0, +\infty) \times [0, +\infty)$.

(H$_2$) There exists $(a, b) \subset [0, 1]$ such that

$$\lim_{x,y\to+\infty} \min_{t\in[a,b]} \frac{f(t,x,y)+p_1(t)}{y} = +\infty, \quad \text{or}$$

$$\lim_{x,y\to+\infty} \min_{t\in[a,b]} \frac{g(t,x,y)+p_1(t)}{y} = +\infty.$$

(H$_3$) Assume that

$$\int_0^1 k(s)\,\mathrm{d}s < \frac{\Gamma(\alpha)\alpha^\alpha r}{4(\alpha-1)(\alpha-2)^{\alpha-1}M},$$

where

$$M = \max_{x,y\in[0,r]} h(x,y), \quad r_1 = \int_0^1 p_1(s)\,\mathrm{d}s, \quad r_2 = \int_0^1 p_2(s)\,\mathrm{d}s,$$

$$r = \max\left\{ \frac{(\alpha-1)^2(\alpha-2)r_i}{c_i\Gamma(\alpha)}, i = 1, 2. \right\}.$$

□

Lemma 2.6. *For functions $p_i(t), i = 1, 2$ in* (H$_1$)*, then the boundary value problem*

$$\begin{aligned} &D_{0+}^\alpha u(t) + p_i(t) = 0, \quad 0 < t < 1, \\ &u(0) = u(1) = u'(0) = u'(1) = 0, \end{aligned} \tag{4}$$

has a unique solution $w_i(t) = \int_0^1 G(t,s)p_i(s)\,\mathrm{d}s$ with

$$w_i(t) \leq (\alpha-1)q(1-t)\int_0^1 p_i(s)\,\mathrm{d}s, t \in [0,1], \quad i = 1, 2. \tag{5}$$

Proof. By Lemma 2.3 and Lemma 2.4, we have $w_i(t) = \int_0^1 G(t,s)p_i(s)\,\mathrm{d}s$ is the unique solution of (3) and

$$w_i(t) = \int_0^1 G(t,s)p_i(s)\,\mathrm{d}s \leq (\alpha-1)q(1-t)\int_0^1 p_i(s)\,\mathrm{d}s, \quad i = 1, 2.$$

For any $x \in C[0,1]$, we define a function $[x(\cdot)]^* : [0,1] \to [0,+\infty)$ by

$$[x(\cdot)]^* = \begin{cases} x(t), & x(t) \geq 0, \\ 0, & x(t) < 0. \end{cases}$$

In order to overcome the difficulty associated with semipositone, we consider the following approximately singular nonlinear differential system:

$$\begin{aligned} &D_{0+}^{\alpha}u(t) + f\left(t, [u(t) - w_1(t)]^*, [v(t) - w_2(t)]^*\right) + p_1(t) = 0, \quad 0 < t < 1, \\ &D_{0+}^{\alpha}v(t) + g\left(t, [u(t) - w_1(t)]^*, [v(t) - w_2(t)]^*\right) + p_2(t) = 0, \quad 0 < t < 1, \\ &u(0) = u(1) = u'(0) = u'(1) = v(0) = v(1) = v'(0) = v'(1) = 0, \end{aligned} \tag{6}$$

where $w_i(t)$ $(i = 1, 2)$ are defined in Lemma 2.6.

It is well-known that the problem (6) can be written equivalently as the following nonlinear system of integral equations

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)\left[f\left(s, [u(s) - w_1(s)]^*, [v(s) - w_2(s)]^*\right) + p_1(t)\right] \mathrm{d}s, \quad 0 \le t \le 1, \\ v(t) &= \int_0^1 G(t,s)\left[g\left(s, [u(s) - w_1(s)]^*, [v(s) - w_2(s)]^*\right) + p_2(t)\right] \mathrm{d}s, \quad 0 \le t \le 1. \end{aligned} \tag{7}$$

We consider the Banach space $X = C[0,1]$ with the norm $\|u\| = \max_{0\le t\le 1} |u(t)|$, and the Banach space $Y = X \times X$ with the norm $\|(u,v)\| = \max\{\|u\|, \|v\|\}$.

We define the cone $P \subset Y$ by

$$P = \left\{ (u,v) \in Y | u(t) \ge \frac{c_1 t^{\alpha-2}(1-t)^2}{\alpha - 1} \|(u,v)\|, v(t) \ge \frac{c_2 t^{\alpha-2}(1-t)^2}{\alpha - 1} \|(u,v)\|, t \in [0,1] \right\}.$$

Define the operators $T_1, T_2 : Y \to X$ and $T : Y \to Y$ as follows:

$$\begin{aligned} T_1(u,v)(t) &= \int_0^1 G(t,s)\left[f\left(s, [u(s) - w_1(s)]^*, [v(s) - w_2(s)]^*\right) + p_1(t)\right] \mathrm{d}s, \quad 0 \le t \le 1, \\ T_2(u,v)(t) &= \int_0^1 G(t,s)\left[g\left(s, [u(s) - w_1(s)]^*, [v(s) - w_2(s)]^*\right) + p_2(t)\right] \mathrm{d}s, \quad 0 \le t \le 1, \end{aligned}$$

and $T(u,v) = \left(T_1(u,v), T_2(u,v)\right), (u,v) \in Y$. Thus, the solutions of our problem (6) are the fixed points of the operator T. □

Lemma 2.7 (see [5]). *Let E be a real Banach space, P be a cone in E. Ω be a bounded open subset of E with $\theta \in \Omega$, and $T : \overline{\Omega} \cap P \to P$ be a completely continuous operator, then the following conclusions hold:*

(i) *Suppose that $Tu \ne \lambda u, \forall u \in \partial\Omega \cap P, \lambda \ge 1$, then $i(T, \Omega \cap P, P) = 1$.*

(ii) *Suppose that $Tu \nleq u, \forall u \in \partial\Omega \cap P$, then $i(T, \Omega \cap P, P) = 0$.*

3. Main Results and Proof

Lemma 3.1. *$T: P \to P$ is a completely continuous operator.*

Proof. Let $(u, v) \in P$ be an arbitrary element. From Lemma 2.4 and (H_1), we can get

$$\begin{aligned}\left\|T_1(u,v)\right\| &= \max_{0\le t\le 1}\left|T_1(u,v)(t)\right| \\ &\le \int_0^1 (\alpha-1)q(s)\left[f\left(s,[u(s)-w_1(s)]^*,[v(s)-w_2(s)]^*\right)+p_1(t)\right]\,\mathrm{d}s \\ &\le (\alpha-1)\int_0^1 q(s)k(s)h\left([u(s)-w_1(s)]^*,[v(s)-w_2(s)]^*\right)\,\mathrm{d}s, \\ \left\|T_2(u,v)\right\| &= \max_{0\le t\le 1}\left|T_2(u,v)(t)\right| \\ &\le \int_0^1 (\alpha-1)q(s)\left[g\left(s,[u(s)-w_1(s)]^*,[v(s)-w_2(s)]^*\right)+p_2(t)\right]\,\mathrm{d}s \\ &\le (\alpha-1)\int_0^1 q(s)k(s)h\left([u(s)-w_1(s)]^*,[v(s)-w_2(s)]^*\right)\,\mathrm{d}s,\end{aligned}$$

Hence, we obtain

$$\|T(u,v)\| \le (\alpha-1)\int_0^1 q(s)\left[k(s)h\left([u(s)-w_1(s)]^*,[v(s)-w_2(s)]^*\right)\right]\,\mathrm{d}s. \tag{8}$$

Applying (H_1) and (8), we have

$$\begin{aligned}T_1(u,v)(t) &\ge t^{\alpha-2}(1-t)^2\int_0^1 q(s)\left[f\left(s,[u(s)-w_1(s)]^*,[v(s)-w_2(s)]^*\right)+p_1(t)\right]\,\mathrm{d}s \\ &\ge t^{\alpha-2}(1-t)^2\int_0^1 q(s)a_1(s)h\left([u(s)-w_1(s)]^*,[v(s)-w_2(s)]^*\right)\,\mathrm{d}s \\ &\ge c_1t^{\alpha-2}(1-t)^2\int_0^1 q(s)k(s)h\left([u(s)-w_1(s)]^*,[v(s)-w_2(s)]^*\right)\,\mathrm{d}s \\ &\ge \frac{c_1t^{\alpha-2}(1-t)^2}{\alpha-1}\|T(u,v)\|.\end{aligned}$$

In the similar manner, we deduce

$$T_2(u,v)(t) \ge \frac{c_2t^{\alpha-2}(1-t)^2}{\alpha-1}\|T(u,v)\|.$$

Thus $T(u, v) \in P$, that is $T(P) \subset P$.

According to the Arzela-Ascoli theorem, we can easily get that $T: P \to P$ is a completely continuous operator. □

Theorem 3.1. *If (H_1)–(H_3) hold, then the boundary value problem (1) has at least one positive solution.*

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Proof. Let $P_r = \{(u,v) \in P, \|(u,v)\| < r\}$, we first prove $T(u,v) \neq \lambda(u,v)$, for $\forall (u,v) \in \partial P_r, \lambda \geq 1$, where the constant r is defined in (H_3).

In fact, if not, there exist $\lambda_0 \geq 1$ and $(u,v) \in \partial P_r$ such that $\lambda_0(u,v) = T(u,v)$, then $(u,v) = \frac{1}{\lambda_0}T(u,v)$ and $0 < \frac{1}{\lambda_0} \leq 1$. Since

$$u(t) \geq \frac{c_1 t^{\alpha-2}(1-t)^2}{\alpha-1}\|(u,v)\| = \frac{c_1 t^{\alpha-2}(1-t)^2}{\alpha-1}r, \quad t \in [0,1],$$

$$v(t) \geq \frac{c_2 t^{\alpha-2}(1-t)^2}{\alpha-1}\|(u,v)\| = \frac{c_2 t^{\alpha-2}(1-t)^2}{\alpha-1}r, \quad t \in [0,1],$$

and

$$w_1(t) \leq (\alpha-1)q(1-t)\int_0^1 p_1(s)\,\mathrm{d}s \leq (\alpha-1)q(1-t)r_1,$$

$$w_2(t) \leq (\alpha-1)q(1-t)\int_0^1 p_2(s)\,\mathrm{d}s \leq (\alpha-1)q(1-t)r_2,$$

for any $t \in [0,1]$, we get that

$$\begin{aligned} u(t) - w_1(t) &\geq \frac{c_1 t^{\alpha-2}(1-t)^2}{\alpha-1}r - (\alpha-1)q(1-t)r_1 \\ &\geq \left[\frac{c_1\Gamma(\alpha)r}{(\alpha-1)(\alpha-2)} - (\alpha-1)r_1\right]q(1-t) \geq 0, \\ v(t) - w_2(t) &\geq \frac{c_2 t^{\alpha-2}(1-t)^2}{\alpha-1}r - (\alpha-1)q(1-t)r_2 \\ &\geq \left[\frac{c_2\Gamma(\alpha)r}{(\alpha-1)(\alpha-2)} - (\alpha-1)r_2\right]q(1-t) \geq 0. \end{aligned}$$

Hence by $(u,v) = \frac{1}{\lambda_0}T(u,v)$, we obtain that

$$\begin{aligned} u(t) &= \frac{1}{\lambda_0}\int_0^1 G(t,s)\left[f\left(s, u(s) - w_1(s), v(s) - w_2(s)\right) + p_1(s)\right]\,\mathrm{d}s \\ &\leq \int_0^1 G(t,s)k(s)h\left(s, u(s) - w_1(s), v(s) - w_2(s)\right)\,\mathrm{d}s \\ &\leq (\alpha-1)q(1-t)\int_0^1 k(s)h\left(s, u(s) - w_1(s), v(s) - w_2(s)\right)\,\mathrm{d}s \\ &\leq (\alpha-1)q(1-t)M\int_0^1 k(s)\,\mathrm{d}s. \end{aligned}$$

Since $(u,v) \in \partial P_r$, we know $\|u\| = r$ or $\|v\| = r$. If $\|u\| = r$, then from Lemma 2.5, we deduce

$$\begin{aligned} r = \max_{t\in[0,1]} u(t) &\leq \max_{t\in[0,1]}\left\{(\alpha-1)q(1-t)M\int_0^1 k(s)\,\mathrm{d}s\right\} \\ &\leq \frac{4(\alpha-1)(\alpha-2)^{\alpha-1}M}{\Gamma(\alpha)\alpha^\alpha}\int_0^1 k(s)\,\mathrm{d}s. \end{aligned}$$

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Consequently

$$\int_0^1 k(s)\,\mathrm{d}s \geq \frac{\Gamma(\alpha)\alpha^\alpha r}{4(\alpha-1)(\alpha-2)^{\alpha-1}M},$$

which is a contradiction to (H_3). The proof will be similar when $\|v\| = r$. Therefore, applying Lemma 2.7, we obtain $i\left(T, P_r, P\right) = 1$.

On the other hand, choose a constant $L > 0$ such that

$$L > \frac{(\alpha-1)\alpha^\alpha}{2c_2 a^{\alpha-2}(1-b)^2(\alpha-2)^{\alpha-2}\int_a^b q(s)\,\mathrm{d}s}. \tag{9}$$

From (H_2), there exists $R_1 > r$ such that

$$f(t,x,y) + p_1(t) \geq Ly, \quad \forall t \in [a,b], x, y \geq R_1. \tag{10}$$

Taking $R \geq 2R_1 \max\left\{\frac{\alpha-1}{c_i a^{\alpha-2}(1-b)^2}, i = 1, 2\right\}$. Obviously, $R > 2R_1 > 2r$, thus $\frac{r}{R} < \frac{1}{2}$.

Let $P_R = \{(u,v) \in P, \|(u,v)\| < R\}$, we will show that $T(u,v) \nleq (u,v)$, for $\forall (u,v) \in \partial P_R$.

In fact, otherwise, there exists $(u,v) \in \partial P_R$ such that $T(u,v) \leq (u,v)$. By proceeding as for the proof to get (9) and (H_3), we have, for any $t \in [a,b]$,

$$\begin{aligned}
u(t) - w_1(t) &\geq u(t) - (\alpha-1)q(1-t)r_1 \geq u(t) - \frac{c_1\Gamma(\alpha)r}{(\alpha-1)(\alpha-2)}q(1-t) \\
&= u(t) - \frac{c_1 t^{\alpha-2}(1-t)^2}{\alpha-1}r \geq u(t) - \frac{u(t)}{R}r \geq \frac{1}{2}u(t) \\
&\geq \frac{1}{2}R\frac{c_1 t^{\alpha-2}(1-t)^2}{\alpha-1} \geq \frac{1}{2}R\frac{c_1 a^{\alpha-2}(1-b)^2}{\alpha-1} \geq R_1 > 0, \\
v(t) - w_2(t) &\geq v(t) - (\alpha-1)q(1-t)r_2 \geq v(t) - \frac{c_2\Gamma(\alpha)r}{(\alpha-1)(\alpha-2)}q(1-t) \\
&= v(t) - \frac{c_2 t^{\alpha-2}(1-t)^2}{\alpha-1}r \geq v(t) - \frac{v(t)}{R}r \geq \frac{1}{2}v(t) \\
&\geq \frac{1}{2}R\frac{c_2 t^{\alpha-2}(1-t)^2}{\alpha-1} \geq \frac{1}{2}R\frac{c_2 a^{\alpha-2}(1-b)^2}{\alpha-1} \geq R_1 > 0.
\end{aligned}$$

Therefore, we deduce

$$\begin{aligned}
R \geq u(t) \geq T_1(u,v)(t) &= \int_0^1 G(t,s)\left[f\left(s, [u(s)-w_1(s)]^*, [v(s)-w_2(s)]^*\right) + p_1(t)\right]\,\mathrm{d}s \\
&\geq \int_a^b G(t,s)\left[f\left(s, u(s)-w_1(s), v(s)-w_2(s)\right) + p_1(t)\right]\,\mathrm{d}s \\
&\geq L\int_a^b G(t,s)\left(v(s)-w_2(s)\right)\,\mathrm{d}s \\
&\geq \frac{1}{2}LR\frac{c_2 a^{\alpha-2}(1-b)^2}{\alpha-1}\int_a^b G(t,s)\,\mathrm{d}s \\
&\geq \frac{1}{2}LR\frac{c_2 a^{\alpha-2}(1-b)^2}{\alpha-1}t^{\alpha-2}(1-t)^2\int_a^b q(s)\,\mathrm{d}s, \quad \forall t \in [0,1].
\end{aligned}$$

Then from Lemma 2.5, we have

$$\begin{aligned} R &\geq \frac{1}{2} LR \frac{c_2 a^{\alpha-2}(1-b)^2}{\alpha-1} \int_a^b q(s)\,\mathrm{d}s \max_{t\in[0,1]} \left\{ t^{\alpha-2}(1-t)^2 \right\} \\ &\geq 2LR \frac{c_2 a^{\alpha-2}(1-b)^2(\alpha-2)^{\alpha-2}}{(\alpha-1)\alpha^{\alpha}} \int_a^b q(s)\,\mathrm{d}s. \end{aligned}$$

Consequently, by (10) we obtain

$$R \geq 2LR \frac{c_2 a^{\alpha-2}(1-b)^2(\alpha-2)^{\alpha-2}}{(\alpha-1)\alpha^{\alpha}} \int_a^b q(s)\,\mathrm{d}s > R.$$

This is a contradiction. Thus from Lemma 2.7, we get $i\left(T, P_R, P\right) = 0$.

From the properties of the fixed point index, we have $i\left(T, P_R \setminus P_r, P\right) = -1$. Therefore, T has a fixed point (u_0, v_0) in $P_R \setminus \overline{P}_r$, with $\left\|(u_0, v_0)\right\| > r$. At the same time,

$$\begin{aligned} u_0(t) - w_1(t) &\geq \frac{c_1 t^{\alpha-2}(1-t)^2}{\alpha-1} \left\|(u_0, v_0)\right\| - (\alpha-1)q(1-t)r_1 \\ &> \left[\frac{c_1 \Gamma(\alpha) r}{(\alpha-1)(\alpha-2)} - (\alpha-1)r_1 \right] q(1-t) \geq 0, \\ v_0(t) - w_2(t) &\geq \frac{c_2 t^{\alpha-2}(1-t)^2}{\alpha-1} \left\|(u_0, v_0)\right\| - (\alpha-1)q(1-t)r_2 \\ &> \left[\frac{c_2 \Gamma(\alpha) r}{(\alpha-1)(\alpha-2)} - (\alpha-1)r_2 \right] q(1-t) \geq 0. \end{aligned}$$

Then, we known that $(u_0(t), v_0(t))$ is a solution of system (6) and $w_i(t)(i = 1, 2)$ are solutions of system (3). Thus $(u_0(t)-w_1(t), v_0(t)-w_2(t))$ is a positive solution of the singular semipositone boundary value problem (1).

The proof of Theorem 3.1 is completed. □

Remark 3.1. The conclusion of Theorem 3.1 is valid if (H_2) is replaced by

$(\mathrm{H}_2)^*$ There exists $(a, b) \subset [0, 1]$ such that

$$\begin{aligned} &\lim_{x,y\to+\infty} \min_{t\in[a,b]} \frac{f(t, x, y) + p_1(t)}{x} \geq \overline{L}, \text{ or} \\ &\lim_{x,y\to+\infty} \min_{t\in[a,b]} \frac{g(t, x, y) + p_1(t)}{x} \geq \overline{L}, \end{aligned}$$

where

$$\overline{L} > \frac{(\alpha-1)\alpha^{\alpha}}{2c_1 a^{\alpha-2}(1-b)^2(\alpha-2)^{\alpha-2} \int_a^b q(s)\,\mathrm{d}s}.$$

4. Example

Now, we present an example to illustrate the main results.

Example 4.1. Consider the following system of fractional differential equations

$$\begin{aligned} &D_{0+}^{\frac{7}{2}}u(t) + \frac{1}{10}t^{-\frac{1}{5}}\left(|u(t)|^2 + |v(t)|^2\right) - \frac{1}{16}t^{-\frac{1}{8}} = 0, \quad 0 < t < 1, \\ &D_{0+}^{\frac{7}{2}}v(t) + \frac{1}{5}t^{-\frac{1}{5}}\left(|u(t)|^2 + |v(t)|^2\right) - \frac{1}{4}t^{-\frac{1}{4}} = 0, \quad 0 < t < 1, \\ &u(0) = u(1) = u'(0) = u'(1) = v(0) = v(1) = v'(0) = v'(1) = 0. \end{aligned} \tag{11}$$

In the system (11), $\alpha = \frac{7}{2}$ and

$$f(t,u,v) = \frac{1}{10}t^{-\frac{1}{5}}\left(|u(t)|^2 + |v(t)|^2\right) - \frac{1}{16}t^{-\frac{1}{8}},$$

$$g(t,u,v) = \frac{1}{5}t^{-\frac{1}{5}}\left(|u(t)|^2 + |v(t)|^2\right) - \frac{1}{4}t^{-\frac{1}{4}},$$

for $t \in [0,1], u, v \geq 0$.

We deduce $p_1(t) = \frac{1}{16}t^{-\frac{1}{8}}, p_2(t) = \frac{1}{4}t^{-\frac{1}{4}}, k(t) = \frac{1}{5}t^{-\frac{1}{5}}, a_i(t) = \frac{1}{15}t^{-\frac{1}{5}}, c_i = \frac{1}{3}, i = 1,2$. $h(u,v) = |u(t)|^2 + |v(t)|^2$.

Clearly, f, g satisfy conditions (H_1) and (H_2). Since

$$r_1 = \int_0^1 p_1(s)\,ds = \frac{1}{14}, \quad r_2 = \int_0^1 p_2(s)\,ds = \frac{1}{3}, \quad \int_0^1 k(s)\,ds = \frac{1}{4}.$$

We have $r = \max\left\{\frac{(\alpha-1)^2(\alpha-2)r_i}{c_i\Gamma(\alpha)}, i = 1,2\right\} = \frac{5}{\sqrt{\pi}}$ and consequently

$$M = \frac{50}{\pi}, \quad \frac{\Gamma(\alpha)\alpha^\alpha r}{4(\alpha-1)(\alpha-2)^{\alpha-1}M} \approx 1.71.$$

So

$$\int_0^1 k(s)\,ds < \frac{\Gamma(\alpha)\alpha^\alpha r}{4(\alpha-1)(\alpha-2)^{\alpha-1}M}.$$

It is thus clear that (H_3) is satisfied. Hence it follows from Theorem 3.1 that system (11) has at least one positive solution.

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Competing Interests

The authors declare no competing interests.

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