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Research Article

Hopf Bifurcation in a Delayed Solow-Verhulst Model

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Abstract. In this paper, we propose a mathematical study of the relationship between population dynamics and economic growth. To do this, the total population is divided into three disjoint classes: employed, unemployed and economically inactive population. On the one hand, the evolution of the number of individuals in each compartment is described by Verhulst model and on the other hand the economic growth is governed by the Solow equation. The resulting model is a system of differential equations with time delay. The dynamics, of this system, are studied in terms of local stability and of local Hopf bifurcation. Some numerical simulations are given to illustrate our theoretical results. Additionally we conclude with some remarks.

Keywords: Solow growth model; Verhulst model; delay differential equations; Hopf bifurcation.

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1. Introduction

The relationship between population dynamics and economic growth has long attracted the attention of researchers in economics and demography. These empirical studies have led to conflicting conclusions. Faced with these controversies, some researchers in applied mathematics focused their magnifying glasses in order to analyze theoretically this problematic. These researches have concluded, in this case a very complex relationship between population and economic growth.

In this work, we follow the ideas in [3, 10, 16, 22], to model the interaction between the population dynamic and economic growth. We divide the total population into three disjoint classes: employed population (all persons who perform some work for pay or profit), unemployed population (all persons seeking work) and economically inactive population (attendant at educational institutions, retired, engaged in family duties and other economically inactive). On the one hand, we describe the evolution of the number of individuals in each compartment by Verhulst model and on the other hand we combine the resulting equations with the Solow model of economic growth. In this way, we obtain the following delay differential system:

$$\begin{aligned} \frac{dK}{dt} &= sf(K(t), L(t)) - \delta K(t), \\ \frac{dL}{dt} &= \gamma p N(t)\left[1 - \frac{L(t-\tau)}{L_e}\right], \\ \frac{dN}{dt} &= \rho N(t)\left[1 - \frac{N(t)}{g(K(t), L(t))}\right], \end{aligned} \tag{1}$$

where K is the capital stock, L is the size of the employed population, N is the total number of population, p is the unemployed rate, s is the saving rate, δ is the depreciation rate of capital stock, γ is the employment rate, ρ is the population growth rate, f is the production function, L_e is the effective labor demand (the total demand for employees in the labor market) [10, 16], g is the carrying capacity and τ is the time needed to assess needs for labor force and the time taken for the recruitment of this labor force [12].

The first model in this optic is presented by Slow and Swan [22, 23]. This model is gouverned by the following ordinary differential equation:

$$\frac{dK}{dt} = sf(K(t), L_0 e^{nt}), \tag{2}$$

here the author has assumed that the function f is with constant returns to scale, the growth rate of labor n and the savings rate s are constant. The contribution of the model (2) is to show the effect of the capital stock and the labor force on the total output of a nation.

In (2006, [15]), Guerrini has considered the SolowSwan growth model with nonconstant labor growth rate as follows:

$$\frac{dk}{dt} = sf(k(t)) - (\delta + n(t))k(t), \tag{3}$$

where $k = \frac{K}{L}$ is the capital-labor ratio (per capita capital), δ is the depreciation rate of capital stock, and $n(t)$ is the population growth rate.

In his conclusion, the author has observed that the per capita capital of a country will tend to stabilize to the non-trivial steady state of the system (3), independently of its initial value. In addition, he reported that two countries with same initial per capita capital and with two different labor growth rate will stabilize to the same per capita capital if the limits of their labor growth rates are equal.

In (2010, [3]), Cai proposed an economic growth model with endogenous carrying capacity as follows:

$$\begin{aligned} \frac{dK}{dt} &= sf(K, L) - \delta K, \\ \frac{dL}{dt} &= \gamma L \left[1 - \frac{L}{g(K)}\right], \end{aligned} \tag{4}$$

In his work, Cai has proved that the dynamical system (4) has one, two or three equilibria under different conditions and that this system undergoes a saddle-node bifurcations and the associated economy has multiple growth paths under the specified parameters. The author has observed the "Malthusian trap" which appears in the economy when the technological level and the saving rate are low or the carrying capacity grows slowly. However, the economy can escape the "Malthusian trap" by promoting technological levels or the saving rate or having high carrying capacity growth rate. In (2012, [2]), the author has proved that if the system (4) has a unique non trivial equilibrium, then it is asymptotically stable. Furthermore, he showed, by qualitative analysis, that the demographic transition appears under the interaction between economic growth and human population carrying capacity.

In (2013, [14]), Guerrini and Sodini have introduced the same time delay in the production function and in the logistic equation of the evolution of the working population. The resulting model is the following delayed differential system:

$$\begin{aligned} \frac{dK}{dt} &= sf(K(t-\tau), L) - \delta K, \\ \frac{dL}{dt} &= \gamma L[a - bL(t-\tau)], \end{aligned} \tag{5}$$

Here, the authors have considered the logistic equation with constant carrying capacity and they showed that the time delay is a source of cyclical behavior in the model (5).

In (2013, [13]), Guerrini proposed the following generalization of the model (4):

$$\begin{aligned} \frac{dK}{dt} &= sf(K(t-\tau), L(t)) - \delta K(t-\tau), \\ \frac{dL}{dt} &= \gamma L \left[1 - \frac{L(t)}{g(K(t))}\right], \end{aligned} \tag{6}$$

In this model, Guerrini introduced the time delay in production function and in capital depreciation function. He assumed that there is full employment in the economy, so that employment and labor supply coincide, i.e. L = N. He proved that the system (6) loses stability and a Hopf bifurcation occurs when the time delay passes through critical values.

Recently, we proposed the following model of mutual interactions between the economically active population and the economic growth [12]:

$$\begin{aligned} \frac{dK}{dt} &= sf(K(t), L(t)) - \delta K(t), \\ \frac{dL}{dt} &= \gamma L(t) \left[1 - \frac{L(t-\tau)}{g_1(K(t-\tau))}\right]. \end{aligned} \tag{7}$$

We have assumed that economy is not at full employment ($L < N$) and they showed that a time delay in recruitment process can destabilize the system (7) giving birth to economic fluctuations.

In all the aforementioned works, the authors concentrate on the relationship between employed population and economic growth. Our main contribution in this paper is to extend the analysis of this relationship to include the effect of total population and unemployed population. To do this, we study the dynamics of the system (1) in terms of local stability and of the description of the Hopf bifurcation, that is proven to exist as the delay cross some critical values.

2. Local Stability and Hopf Bifurcation Analysis

We assume that

(H_1): the functions g is twice continuously differentiable,

(H_2): $f(K, L) = AK^{\alpha}L^{1-\alpha}$ (see, [5, 22]),

where A is a positive constant that reflects the level of the technology and $\alpha \in (0, 1)$ is the elasticity between capital and labor.

2.1. Equilibria

Proposition 1. *The system* (1) *has two equilibria: the trivial equilibrium* $E_0 = (0, 0, 0)$ *and the nontrivial equilibrium* $E_1 = (K_*, L_e, g(K_*, L_e))$*, where* $K_* = (\frac{sA}{\delta})^{\frac{1}{1-\alpha}} L_e$.

Proof. To find equilibria, we consider the following system:

$$\begin{aligned} sAK^{\alpha}L^{1-\alpha} - \delta K = 0, \\ N\left[1 - \frac{L}{L_e}\right] = 0, \\ N\left[1 - \frac{N}{g(K, L)}\right] = 0, \\ L(t) \leq N(t). \end{aligned} \tag{8}$$

It is easy to verify that $(0, 0, 0)$ is a solution of the system (8), thus $E_0 = (0, 0, 0)$ is a trivial equilibrium of the system (1). In the following, we prove that $E_1 = (K_*, L_e, g(K_*, L_e))$ is the unique nontrivial equilibrium of the system (1).

(K_*, L_*, N_*) is a positive solution of the system (8) if and only if

$$\begin{aligned} sAK_*^{\alpha}L_e^{1-\alpha} - \delta K_* = 0, \\ L_* = L_e, \\ N_* = g(K_*, L_*). \end{aligned} \tag{9}$$

It's clear that the first equation of (9) has a unique solution $K_* = (\frac{sA}{\delta})^{\frac{1}{1-\alpha}} L_e$. This concludes the proof. □

2.2. Stability and Hopf bifurcation analysis

By analyzing the characteristic equation associated to (1), we determine necessary conditions for the linear stability and Hopf bifurcation around the equilibrium E_1.

Consider the change of variables $x = K - K_*$, $y = L - L_*$ and $z = N - N_*$. Then by linearizing system (1) around (K_*, L_*, N_*) we have

$$\begin{aligned} \frac{dx}{dt} = (\alpha - 1)\delta x + s\frac{\partial f(K_*, L_*)}{\partial L}y, \\ \frac{dy}{dt} = -\gamma p y_{\tau}, \\ \frac{dz}{dt} = \rho\frac{\partial g(K_*, L_*)}{\partial K}x + \rho\frac{\partial g(K_*, L_*)}{\partial L}y - \rho z. \end{aligned} \tag{10}$$

The characteristic equation associated to system (10) takes the form:

$$\lambda^3 + A\lambda^2 + B\lambda + C + (D\lambda^2 + E\lambda + F)e^{-\lambda\tau} = 0, \tag{11}$$

where

$$A = (1-\alpha)\delta + \rho, \quad B = (1-\alpha)\delta\rho, \quad C = 0, \quad D = \gamma p,$$

$$E = [(1-\alpha)\delta + \rho]\gamma p, \quad F = (1-\alpha)\delta\rho\gamma p.$$

Before giving our main result, we need the following lemma.

Lemma 2. *Consider the equation:*

$$h(z) := z^3 + az^2 + bz + c = 0. \tag{12}$$

where

$$a = (1-\alpha)^2\delta^2 + \rho^2 - \gamma^2 p^2,$$

$$b = (1-\alpha)^2\delta^2\rho^2 - \gamma^2 p^2[(1-\alpha)^2\delta^2 + \rho^2],$$

$$c = -\gamma^2 p^2 (1-\alpha)^2\delta^2\rho^2.$$

Then $z_0 = \gamma^2 p^2$ *is a simple positive root of the equation* (12).

Proof. It is easy to verify that

$$h(\gamma^2 p^2) = 0$$

and

$$h'(\gamma^2 p^2) = 40\gamma^4 p^4 + 7[(1-\alpha)^2\delta^2\rho^2]\gamma^2 p^2 + (1-\alpha)^2\delta^2\rho^2 > 0.$$

Thus $z_0 = \gamma^2 p^2$ is a simple positive root of the equation (12). □

The objective of the following theorem is to investigate the local stability and the existence of a Hopf bifurcation of the system (1) at the equilibrium E_1.

Theorem 3. *If the hypotheses* (H_1) *and* (H_2) *hold, then there exists a critical positive delay* τ_0 *such that,*

1. *for* $\tau \in [0, \tau_0)$, *the equilibrium* E_1 *of the system* (1) *is locally asymptotically stable;*
2. *for* $\tau > \tau_0$, *the equilibrium* E_1 *of the system* (1) *is unstable;*
3. *for* $\tau = \tau_0$, *a Hopf bifurcation of periodic solutions of system* (1) *occurs at the equilibrium* E_1.

Proof. The equation (11) is equivalent to

$$(\lambda + (1-\alpha)\delta)(\lambda + \rho)(\lambda + \gamma p e^{-\lambda\tau}) = 0. \tag{13}$$

For $\tau = 0$, all roots of equation (13) are strictly negative. Moreover, there exists a critical positive delay τ_0 such that the equation (13) has a simple pair of pure conjugate imaginary roots $\pm\gamma pi$.

Next, we need to calculate τ_0 and to verify the transversally condition.

Let $\lambda(\tau) = u(\tau) + iv(\tau)$ be the root of (11). Thus,

$$u^3 - 3uv^2 + Au^2 - Av^2 + Bu + C = -\exp(-u\tau)\{Du^2\cos(v\tau)$$

$$- Dv^2\cos(v\tau) + Eu\cos(v\tau) + F\cos(v\tau) + 2Duv\sin(v\tau) + Ev\sin(v\tau)\}, \qquad (14)$$

and

$$3u^2v - v^3 + 2Auv + Bv = -\exp(-u\tau)\{2Duv\cos(v\tau)$$

$$+ Ev\cos(v\tau) - Du^2\sin(v\tau) + Dv^2\sin(v\tau) - Eu\sin(v\tau) - F\sin(v\tau)\}. \qquad (15)$$

To calculate τ_0, we set $u(\tau_0) = 0$ and $v(\tau_0) = \gamma p$ into the two equations (14) and (15) to get

$$-A\rho(\gamma p)^2 = (D(\gamma p)^2 - F)\cos(\gamma p\tau_0) - E\gamma p\sin(\gamma p\tau_0), \qquad (16)$$

and

$$(\gamma p)^3 - B\gamma p = E\gamma p\cos(\gamma p\tau_0) + (D(\gamma p)^2 - F)\sin(\gamma p\tau_0). \qquad (17)$$

Solving (16) and (17) simultaneously gives

$$\tau_0 = \frac{1}{\gamma p}\arccos\left(\frac{A(\gamma p)^2(F - D(\gamma p)^2) + ((\gamma p)^3 - B\gamma p)E\gamma p}{(D\gamma p - F)^2 + E^2(\gamma p)^2}\right).$$

To verify the transversally condition, We need to prove

$$\frac{d\ Re\ \lambda(\tau_0)}{d\tau} > 0.$$

Squaring the two formulas (16) and (17), and adding the squares together, we obtain,

$$(\gamma p)^6 + a(\gamma p)^4 + b(\gamma p)^2 + c = 0, \qquad (18)$$

Letting $z_0 = \gamma^2 p^2$, the formula (18) becomes as follows

$$h(z_0) := z_0^3 + az_0^2 + bz_0 + c = 0. \qquad (19)$$

By differentiating (14) and (15) with respect to τ and then setting $\tau = \tau_0$, we obtain

$$G_1\frac{du(\tau_0)}{d\tau} + G_2\frac{dv(\tau_0)}{d\tau} = H_1, \qquad (20)$$

$$-G_2\frac{du(\tau_0)}{d\tau} + G_1\frac{dv(\tau_0)}{d\tau} = H_2, \qquad (21)$$

where

$$G_1 = -3v(\tau_0)^2 + B + (E + Dv(\tau_0)^2\tau_0 - F\tau_0)\cos(v(\tau_0)\tau_0) + (2Dv(\tau_0) - Ev(\tau_0)\tau_0)\sin(v(\tau_0)\tau_0),$$

$$G_2 = -2Av(\tau_0) + (-2Dv(\tau_0) + Ev(\tau_0)\tau_0)\cos(v(\tau_0)\tau_0) + (E + Dv(\tau_0)^2\tau_0 - F\tau_0)\sin(v(\tau_0)\tau_0),$$

$$H_1 = (-Dv(\tau_0)^3 + Fv(\tau_0))\sin(v(\tau_0)\tau_0) - Ev(\tau_0)^2\cos(v(\tau_0)\tau_0),$$

$$H_2 = (-Dv(\tau_0)^3 + Fv(\tau_0))\cos(v(\tau_0)\tau_0) + Ev(\tau_0)^2\sin(v(\tau_0)\tau_0).$$

Solving for $\frac{du(\tau_0)}{d\tau}$ we get

$$\frac{du(\tau_0)}{d\tau} = \frac{G_1 H_1 - G_2 H_2}{G_1^2 + G_2^2}. \tag{22}$$

Therefore, we have

$$\frac{du(\tau_0)}{d\tau} = \frac{\gamma^2 p^2 h'(\gamma^2 p^2)}{G_1^2 + G_2^2}, \tag{23}$$

where h is a cubic function such that $h(\gamma^2 p^2)$ is given by (19).

Thus, from lemma 2, we have the transversally condition:

$$\frac{du(\tau_0)}{d\tau} > 0.$$

This completes the proof. □

3. Numerical Simulations

In this section, we study how the dynamics of the model (1) change when the time delay parameters vary. Let's give the following numerical simulations:

Proposition 4. *If* $\alpha = 0.5$; $s = 0.5$; $A = 2$; $\delta = 0.2$; $\gamma = 0.8$; $p = 0.3$; $\rho = 0.7$; $L_e = 1000$; *and* $g(K, L) = 10000$. *Then system* (1) *have the following positive equilibrium*

$$E_1 = (25000; 1000; 10000).$$

Furthermore, the critical delay and the period of oscillations corresponding to (1) *are* $\tau_0 = 0.6552$ *and* $P_0 = \pi/\gamma p$.

4. Conclusion

In order to study the interaction between the population and the economic growth, we propose a Solow-Verhulst model (1) with time delay in the recruitment of the labor force. The construction of this model is based on the distribution of the population into three compartments (employed population, unemployed population and economically inactive population) and on the combination of the following three models:

1. the model of the evolution of the capital stock (Solow [22]);
2. the model of the evolution of the total population (Verhulst [26]);
3. the model of the evolution of the active population (Hallegatte et al. [10, 16]).

The dynamics of our model (1) are studied in terms of local stability and existence of the Hopf bifurcation. First, we obtained the existence of an unique positive equilibrium, we analyzed its stability and finally we have identified the necessary conditions of the existence of nontrivial periodic solutions which bifurcate from this equilibrium.

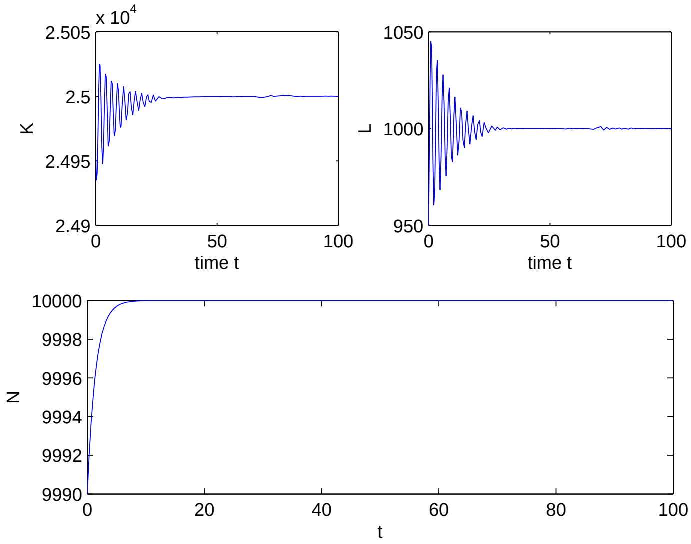

Figure 1: For $\tau = 0.6$, Solution of the model (1) is locally asymptotically stable.

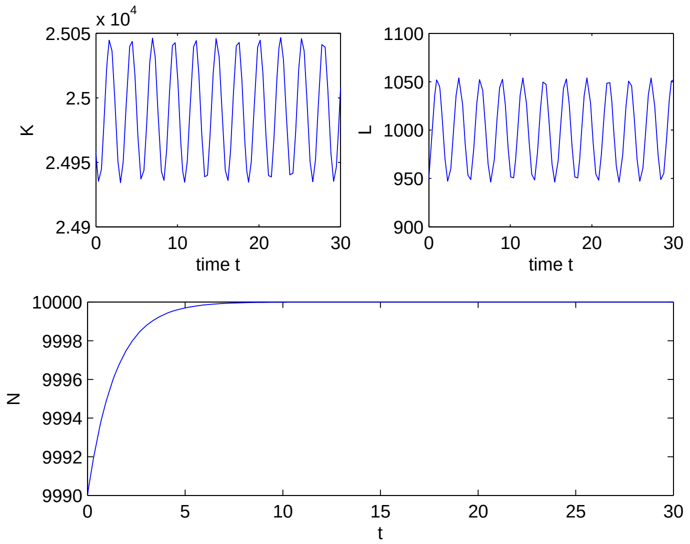

Figure 2: For $\tau = 0.6552$, the model (1) has a semi-periodic solution.

In summary, we found that:

1. The delay in the recruitment can give rise to oscillations in the active population variable.
2. The oscillations in the active population cause the economic fluctuations (oscillations in the capital stock).
3. The total population growth is a neutral factor for the economic growth (see Figure 2).
4. The phase shift between the active population and the capital stock can lead a positive, negative or neutral mutual interaction between economic growth and the population. This justified the empirical results.

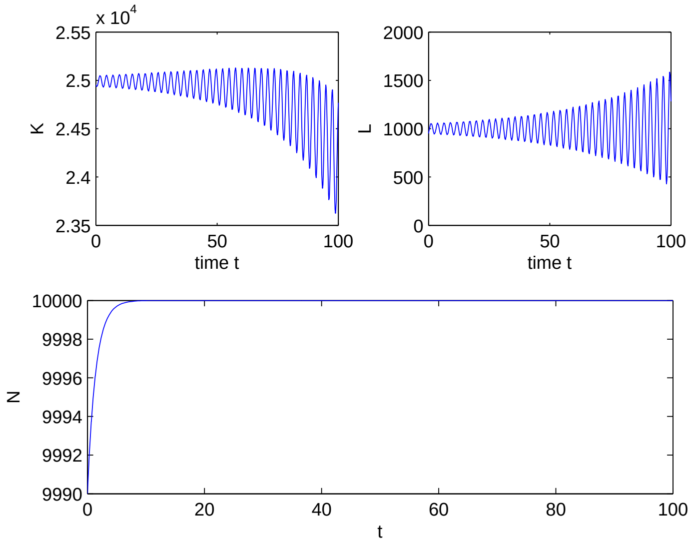

Figure 3: For $\tau = 0.67$, the model (1) is unstable.

Competing Interests

The authors declare that they have no competing interests.

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