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Research Article

Local Convergence for a Frozen Family of Steffensen-Like Methods under Weak Conditions

Ioannis K. Argyros[1] and Santhosh George[2]

[1]*Department of Mathematical Sciences, Cameron University, Lawton, OK 73505, USA*
[2]*Department of Mathematical and Computational Sciences, NIT Karnataka, 575 025, India*

Abstract. In the present paper, we study the local convergence analysis of a Steffensen-like method considered also in Amat et al. [1] modified suitably to solve equations in the Banach space. Using our idea of restricted convergence domains we extend the applicability of this method. Numerical examples where earlier results cannot apply to solve equations but our results can apply are also given in this study.

Corresponding Author

Santhosh George
sgeorge@nitk.ac.in

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1. Introduction

Recently, Amat et al. in [1] studied the efficiency of a frozen family of Steffensen-like methods defined by

$$
\begin{aligned}
x_0^{(0)} &= x_0 \\
x_n^{(j+1)} &= x_n^{(j)} - A_n^{-1} F\left(x_n^{(j)}\right),
\end{aligned}
\tag{1.1}
$$

where $x_{n+1} = x_n^{(k)}, x_n^{(0)} = x_n, x_0 \in D$ an initial point k a natural number, $n = 0, 1, 2, \ldots, 0 \leq j \leq k-1$, $A_n = [x_n, x_n + F(x_n); F]$ with $[.,.] : D \times D \to L(X)$ being a divided difference of order one on D [2]. That is, they considered $k-$step iterative method from the Steffensen's method with frozen divided difference operator for solving a system of nonlinear equations and computed the maximum computational efficiency of the method. In this study we present the local convergence analysis of method (1.1) for approximating the solution of a nonlinear equation

$$
F(x) = 0,
\tag{1.2}
$$

where $F : D \subseteq X \longrightarrow X$ is a continuously Fréchet-differentiable operator and D is a convex subset of the Banach space X. Due to the wide applications, finding solution for the equation (1.2) is an important problem in mathematics.

Our goal is to weaken the assumptions in [1], so that the applicability of the method (1.1) can be extended. Notice that the same technique can be used to extend the applicability of other iterative methods that have appeared in [3–5, 10, 11, 13–15].

The rest of the paper is organized as follows. In Section 2 we present the local convergence analysis. We also provide a radius of convergence, computable error bounds and a uniqueness result. Numerical examples are given in the last section.

2. Local Convergence

The local convergence of method (1.1) is based on some scalar functions and parameters. Let w_0, w, v_0, v, w_1 be continuous, non-negative, non-decreasing functions defined on the interval $[0, +\infty), (0, +\infty), [0, +\infty), (0, +\infty), [0, +\infty)^2$, respectively with values in the interval $[0, +\infty)$ with $w_0(0) = w(0) = w_1(0, 0) = 0$. Define the parameters r_1, r_2 and r_0 by

$$r_1 = \sup\{t \geq 0 : w_0(t) < 1\},$$

$$r_2 = \sup\{t \geq 0 : w_1(t, (1 + v_0(t))t) < 1\}$$

and

$$r_0 = \min\{r_1, r_2\}. \tag{2.1}$$

Define functions g, h and p on the interval $[0, r_0)$ by

$$g(t) = \frac{1}{1 - w_0(t)} \left[\int_0^1 w((1 - \theta)t)d\theta + \frac{(p(t) + w_0(t))v(t)}{1 - p(t)} \right]$$

and

$$h(t) = g(t) - 1,$$

where

$$p(t) = w_1(t, (1 + v_0(t)t).$$

We have that $h(0) = -1 < 0$ and $h(t) \to +\infty$ as $t \to r_0^-$. It then follows from the intermediate value theorem that function h has zeros in the interval $(0, r_0)$. Denote by r the smallest such zero. Then, for each $t \in [0, r)$

$$0 \leq g(t) < 1. \tag{2.2}$$

Let $U(a, \rho), \bar{U}(a, \rho)$ stand respectively for the open and closed balls in X with center $a \in X$ and of radius $\rho > 0$. Next, we present the local convergence analysis of method (1.1) using the preceding notation.

Theorem 2.1. *Let $F : D \subset X \to X$ be continuously Fréchet-differentiable operator with divided differences of order one $[., .; F] : D^2 \to L(X)$. Suppose: there exist $x^* \in D$, and non-decreasing continuous functions w_0, v_0, w_1 defined on the intervals $[0, +\infty), [0, +\infty), [0, +\infty)^2$, respectively with values on the interval $[0, +\infty)$ and $w_0(0) = w_1(0, 0) = 0$ such that for each $x, y \in D$*

$$F(x^*) = 0, \quad F'(x^*)^{-1} \in L(X), \tag{2.3}$$

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq w_0(\|x - x^*\|), \tag{2.4}$$

$$\|F'(x^*)^{-1}([x, y; F] - F'(x^*))\| \leq w_1(\|x - x^*\|, \|y - x^*\|), \tag{2.5}$$

and

$$\|[x, x^*; F]\| \leq v_0(\|x - x^*\|); \tag{2.6}$$

there exist continuous, non decreasing functions w, v defined on the interval $[0, r_0)$ with values on the interval $[0, +\infty)$ and $w(0) = 0$ such that for each $x \in D_0 = D \cap U(x^*, r_0)$

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq w(\|x - y\|), \tag{2.7}$$

$$\|F'(x^*)^{-1}[x, x^*; F]\| \leq v(\|x - y\|), \tag{2.8}$$

and

$$\bar{U}(x^*, R) \subseteq D, \tag{2.9}$$

where the r_0 is defined by (2.1), $R := (1 + v_0(r))r$ and r is defined previously. Then, the sequence $\{x_n\}$ generated for $x_0 \in U(x^*, r) - \{x^*\}$ by method (1.1) is well defined in $U(x^*, r)$, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \ldots$ and converges to x^*. Moreover, the following estimates hold

$$\|x_{n+1} - x^*\| = \|x_n^{(k)} - x^*\| \leq (g(\|x_0 - x^*\|))^{k(n+1)}\|x_0 - x^*\| < r. \tag{2.10}$$

Furthermore, if there exists $R_0 \geq r$ such that

$$w_1(0, R_0) < 1 \quad \text{or} \quad w_1(R_0, 0) < 1, \tag{2.11}$$

then the limit point x^* is the only solution of equation $F(x) = 0$ in $D_1 = D \cap \bar{U}(x^*, R_0)$.

Proof. We shall show using mathematical induction that sequence $\{x_n\}$ satisfies (2.10) and converges to x^*. By hypothesis $x_0 \in U(x^*, r) - \{x^*\}$, (2.4) and the definition of r, we have that

$$\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \leq w_0(\|x_0 - x^*\|) \leq w_0(r_0) < 1. \tag{2.12}$$

It follows from (2.12) and the Banach Lemma on invertible operators [1, 14] that $F'(x)^{-1} \in L(X)$ and

$$\|F'(x_0)^{-1}F'(x^*)\| \leq \frac{1}{1 - w_0(\|x_0 - x^*\|)}. \tag{2.13}$$

We can write by (2.3) that

$$F(x_0) = F(x_0) - F(x^*) = [x_0, x^*; F](x_0 - x^*). \tag{2.14}$$

Then, we have by (2.6) that

$$\|F(x_0)\| \leq v(\|x_0 - x^*\|)\|x_0 - x^*\|. \tag{2.15}$$

We also have that

$$\|x_0 + F(x_0) - x^*\| \leq \|x_0 - x^*\| + \|F(x_0)\|$$

$$< r + v_0(r)r = (1 + v_0(r))r = R,$$

so $x_0 + F(x_0) \in U(x^*, R)$. Next, we show that $A_0^{-1} \in L(X)$. We get by (2.5) and (2.15) that

$$\|F'(x^*)^{-1}(A_0 - F'(x^*))\| \le w_1(\|x_0 - x^*\|, \|x_0 - x^*\| + \|F(x_0)\|)$$

$$\le w_1(\|x_0 - x^*\|, \|x_0 - x^*\| + v_0(\|x_0 - x^*\|)\|x_0 - x^*\|)$$

$$\le w_1(r, r + v_0(r)r) = p(r) < 1, \tag{2.16}$$

so,

$$\|A_0^{-1}F'(x^*)\| \le \frac{1}{1 - p(r)}. \tag{2.17}$$

We also have by method (1.1) that $x_0^{(1)}, x_0^{(2)}, \dots x_0^{(k)} = x_1$ are well defined. Let $j = 0$. We can write

$$x_1^{(0)} - x^* = x_0^{(0)} - x^* - F'(x_0^{(0)})^{-1}F(x_0^{(0)})$$

$$+ A_0^{-1}[(A_0 - F'(x^*)) + (F'(x^*) - F'(x_0^{(0)}))] \tag{2.18}$$

$$[F'(x_0^{(0)})^{-1}F'(x^*)][F'(x^*)^{-1}F(x_0^{(0)})].$$

Using (2.2), (2.3), (2.7), (2.13), (2.17) and (2.18) we obtain, in turn that

$$\begin{aligned} \|x_1^{(0)} - x^*\| &\le \frac{\int_0^1 w((1-\theta)\|x_0^{(0)} - x^*\|)d\theta\|x_0^{(0)} - x^*\|}{1 - w_0(\|x_0^{(0)} - x^*\|)} \\ &+ \frac{(p(\|x_0^{(0)} - x^*\|) + w_0(\|x_0^{(0)} - x^*\|))v(\|x_0^{(0)} - x^*\|)\|x_0^{(0)} - x^*\|}{(1 - w_0(\|x_0^{(0)} - x^*\|))(1 - p(\|x_0^{(0)} - x^*\|))} \\ &= g(\|x_0^{(0)} - x^*\|)\|x_0^{(0)} - x^*\| \le \|x_0^{(0)} - x^*\| < r, \end{aligned} \tag{2.19}$$

which shows (2.10) for $n = 0, k = 0$ and $x_1^{(0)} \in U(x^*, r)$, where we also used that $x_0^{(0)} = x_0$. Similarly, we get that

$$\begin{aligned} \|x_2^{(0)} - x^*\| &\le \frac{\int_0^1 w((1-\theta)\|x_0^{(1)} - x^*\|)d\theta\|x_0^{(1)} - x^*\|}{1 - w_0(\|x_0^{(1)} - x^*\|)} \\ &+ \frac{(p(\|x_0^{(0)} - x^*\|) + w_0(\|x_0^{(1)} - x^*\|))v(\|x_0^{(1)} - x^*\|)\|x_0^{(1)} - x^*\|}{(1 - w_0(\|x_0^{(0)} - x^*\|))(1 - p(\|x_0^{(0)} - x^*\|))} \\ &= g(\|x_0^{(0)} - x^*\|)\|x_0^{(1)} - x^*\| \\ &\le g^2(\|x_0^{(0)} - x^*\|)\|x_0^{(0)} - x^*\| \le \|x_0^{(0)} - x^*\| < r, \end{aligned} \tag{2.20}$$

which shows (2.10) for $k = 1$ and $n = 0$. So, inductively, we obtain that for $0 \le m \le j + 1$

$$\|x_0^{(m+1)} - x^*\| \le g(\|x_0^{(0)} - x^*\|)\|x_0^{(m)} - x^*\|$$

$$\le g^{m+1}(\|x_0^{(0)} - x^*\|)\|x_0^{(0)} - x^*\| \tag{2.21}$$

$$\le \|x_0 - x^*\| < r,$$

which shows (2.10) for $m = 0, 1, 2, \ldots, j + 1, n = 0$ and $x_0^{(m+1)} \in U(x^*, r)$. By simply replacing $x_0^{(0)}, x_0^{(1)}, \ldots x_0^{(k)}$ by $x_i^{(0)}, x_i^{(1)}, \ldots x_i^{(k)}$ in the preceding estimates, we arrive at estimates (2.10). Then, from (2.13), we have the estimate

$$\|x_{n+1} - x^*\| \le c\|x_n - x^*\| < r, \tag{2.22}$$

where $c = g(\|x_0 - x^*\|)^{k(n+1)} \in [0, 1)$, so we deduce that $\lim\limits_{k \to \infty} x_k = x^*$ and $x_{k+1} \in U(x^*, r)$. Finally to show the uniqueness part, let $y^* \in D_1$ with $F(y^*) = 0$. Define $Q = [x^*, y^*; F]$ (or $Q = [y^*, x^*; F]$). Then, using (2.5) and (2.11) we get that

$$
\begin{aligned}
\|F'(x^*)^{-1}(Q - F'(x^*))\| &\le w_1(0, \|x^* - y^*\|) \\
&\le w_1(0, R_0) < 1,
\end{aligned}
\tag{2.23}
$$

so $Q^{-1} \in L(X)$. Then, from the identity $0 = F(y^*) - F(x^*) = Q(y^* - x^*)$, we conclude that $x^* = y^*$. $\square$

Remark 2.2. The sufficient semilocal convergence conditions were given in non-affine invariant form [1]. The local convergence analysis of method (1.1) was studied in [1] based on Taylor expansions and hypotheses reaching up to the third Fréchet derivative of *F*. Moreover, no computable error bounds were given nor the radius of convergence. We have addressed the problems in Theorem 2.1. In order for us to compare the new results with the old ones in [1] we rewrite the conditions in affine invariant form as:

$$\|F'(x^*)^{-1}([x, y; F] - [u, v; F])\| \le K_1(\|x - y\| + \|y - v\|) \tag{2.24}$$

for each $x, y, u, v \in D$ with $x \ne y, u \ne v$. In view of (2.24), we also have that

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \le 2K_2\|x - y\|, \tag{2.25}$$

$$\|F'(x^*)^{-1}([x, y; F] - F'(x^*))\| \le K_3(\|x - x^*\| + \|y - x^*\|) \tag{2.26}$$

and

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \le 2K_4\|x - x^*\|. \tag{2.27}$$

Clearly conditions (2.4, (2.5), (2.7) are weaker than (2.27), (2.26) and (2.25), respectively (see also the numerical examples). Moreover, let $w_0(t) = 2K_0t, w_1(s, t) = K_2s + K_3t$ and $w(t) = 2K_4t$. Then, if $D_0 = D$ and $K_0 = K_2 = K_3 = K_4 = K_1$ then our conditions (2.4), (2.5) and (2.7) reduce to (2.27), (2.26) and (2.25), respectively. Moreover, if D_0 is a strict subset of D, then, we have that

$$K_0 \le K_1$$

$$K_2 \le K_1$$

$$K_3 \le K_1$$

and

$$K_4 \le K_1.$$

Hence, even in this special case the new results are better leading to a wider choice of initial guesses (the new radius of convergence will be at least as large); the error bounds on the distances $\|x_n - x^*\|$ at least as tight (leading to fewer iterations to obtain a desired error tolerance) and an at least as precise information on the location of the solution. Finally, it is also worth noticing that conditions (2.4), (2.5) and (2.7) are weaker than (2.24) used in [1] (in non affine invariant form).

3. Numerical Examples

We present two examples in this section. In the first one, we show that the claims at the end of the Remark 2.2 are justified. In the second example, we show that the results in [1] cannot apply to solve those equations. In both examples, we define for simplicity $[x, y; F] = \frac{1}{2}(F'(x) + F'(y))$ for each $x, y \in D$ with $x \neq y$ and $[x, x; F] = F'(x)$ for each $x \in D$.

Example 3.1. Let $X = Y = \mathbb{R}^3, D = \bar{U}(0, 1), x^* = (0, 0, 0)^T$. Define function F on D for $w = (x, y, z)^T$ by

$$F(w) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

Then, the Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Using the approach in [1] (see also the Remark 2.2), we can choose $\bar{w}_0(t) = et, \bar{w}_1(s, t) = e(s + t), \bar{w}(t) = et, \bar{v}(t) = \bar{v}_0(t) = \frac{e+1}{2}$. Then the radius of convergence is given by

$$\bar{r} = 0.0321$$

Under the new approach, we can choose $w_0(t) = (e-1)t, w_1(s, t) = (e-1)(s+t), v_0(t) = \frac{e+1}{2}, w(t) = 2e^{r_0}t, r_1 = \frac{1}{e-1}, r_2 = \frac{2}{(e-1)(e+5)} = r_0$ and $v(t) = \frac{1+e^{r_0}}{2}$. Then, the radius of convergence r is given by

$$r = 0.5003.$$

Hence, we have that

$$\bar{r} < r.$$

Moreover, we have

$$w_0(t) < \bar{w}_0(t)$$

$$w_1(s, t) < \bar{w}_1(s, t)$$

$$v_0(t) < \bar{v}_0(t)$$

and

$$v(t) < \bar{v}(t).$$

That is the rest of the advantages stated at the Remark 2.2 hold.

Example 3.2. Let $X = C[0, 1]$ and consider the nonlinear integral equation of the mixed Hammerstein-type [1, 2, 6–9, 12] defined by

$$x(s) = \int_0^1 G(s, t)(x(t)^{3/2} + \frac{x(t)^2}{2})dt,$$

where the kernel G is the Green's function defined on the interval $[0, 1] \times [0, 1]$ by

$$G(s, t) = \begin{cases} (1 - s)t, & t \le s \\ s(1 - t), & s \le t. \end{cases}$$

The solution $x^*(s) = 0$ is same as the solution of equation (1.2), where $F : C[0.1] \longrightarrow C[0.1])$ is defined by

$$F(x)(s) = x(s) - \int_0^t G(s, t) \left(x(t)^{3/2} + \frac{x(t)^2}{2} \right) dt.$$

Notice that

$$\left\| \int_0^t G(s, t)dt \right\| \le \frac{1}{8}.$$

Then, we have that

$$F'(x)y(s) = y(s) - \int_0^t G(s, t) \left(\frac{3}{2} x(t)^{1/2} + x(t) \right) dt,$$

so since $F'(x^*(s)) = I$,

$$\| F'(x^*)^{-1}(F'(x) - F'(y)) \| \le \frac{1}{8} \left(\frac{3}{2} \| x - y \|^{1/2} + \| x - y \| \right).$$

Therefore, we can choose

$$w_0(t) = w(t) = \frac{1}{8} \left(\frac{3}{2} t^{1/2} + t \right)$$

and

$$v_0(t) = v(t) = 1 + w_0(t), w_1(s, t) = \frac{1}{2}(w_0(s) + w_0(t)).$$

Then, result in [1] cannot be used to solve this problem, since F' is not Lipschitz. However, our results can apply. Indeed, using the above choice of functions we get that

$$r = 0.3965.$$

Competing Interests

The authors declare no competing interests.

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