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Research Article

Solving Poisson Equation within Local Fractional Derivative Operators

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Abstract. In this paper, we are concerned with finding approximate solutions to local fractional Poisson equation by using the reduced differential transform method (RDTM) and homotopy perturbation transform method (HPTM). The presented methods are considered in the local fractional operator sense. Illustrative examples for handling the local fractional Poisson equation are given. The obtained results are given to show the sample and efficient features of the presented techniques to implement partial differential equations with local fractional derivative operators.

Keywords: Poisson equation; Reduce differential transform method; Yang-Laplace transform; Homotopy perturbation method; Local fractional operator.

Mathematics Subject Classification: 26A33, 34A12, 34A34, 35R11.

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1. Introduction

The Poisson equation plays an important role in mathematical physics [1, 2]; that is, it describes the electrodynamics and intersecting interface [3, 4]. We notice that recently local fractional Poisson equation was analyzed in [5]. Recently, the Poisson equation (PE) with local fractional derivative operators (LFDOs) was presented in [6] as follows:

$$\frac{\partial^{2\alpha} u(x,y)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha} u(x,y)}{\partial y^{2\alpha}} = f(x,y), \quad 0 < \alpha \le 1 \tag{1.1}$$

subject to the initial and boundary conditions

$$\begin{aligned} &u(x,0) = 0, \quad u(x,l) = 0, \\ &u(0,y) = \varphi(y), \quad \frac{\partial^\alpha}{\partial x^\alpha} u(0,y) = \phi(y), \end{aligned} \tag{1.2}$$

where $u(x,y)$ is an unknown function, $\varphi(y)$ and $\phi(y)$ are given functions, and the local fractional derivative operators (LFDOs) of $u(x)$ of order α at $x = x_0$ are given by

$$u^{(\alpha)}(x_0) = \lim_{x \to x_0} \frac{\Delta^\alpha (u(x) - u(x_0))}{(x - x_0)^\alpha}, \tag{1.3}$$

where $\Delta^\alpha (u(x) - u(x_0)) \cong \Gamma(\alpha+1)\,(u(x) - u(x_0))$.

In recent years, a many of approximate and analytical methods have been utilized to solve the ordinary and partial differential equations with local fractional derivative operators such as local fractional Adomian decomposition method [7–11], local fractional variational iteration method [6, 7, 12–15], local fractional function decomposition method [8, 16], local fractional series expansion method [11, 17], local fractional Laplace decomposition method [18, 19], local fractional Laplace variational iteration method [20–23], local fractional homotopy perturbation method [24], local fractional reduce differential transform method [24], local fractional differential transform method [26, 27], and local fractional Laplace transform method [28]. Our main purpose of the paper is to utilize the local fractional RDTM and local fractional HPTM to solve the PE with LFDOs.

2. Analysis of the Methods

Let us consider the following partial differential equation with local fractional derivative operators:

$$L_\alpha u(x,y) + R_\alpha u(x,y) = f(x,y), \quad 0 < \alpha \leq 1, \tag{2.1}$$

where $L_\alpha = \frac{\partial^{n\alpha}}{\partial x^{n\alpha}}$, R_α is a linear local fractional operator, and $f(x,y)$ is the source term.

I. Local Fractional HPTM

The local fractional homotopy perturbation method has been developed and applied to solve a class of local fractional partial differential equations by Yang et al. in 2015 [25]. Based on it, we suggest a new analytical method.

Applying the Yang-Laplace transform on both sides of (2.1), we get

$$Ł_\alpha \left\{L_\alpha u(x,y)\right\} + Ł_\alpha \left\{R_\alpha u(x,y)\right\} = Ł_\alpha \left\{f(x,y)\right\}. \tag{2.2}$$

Using the property of the Yang-Laplace transform, we have

$$\begin{aligned} s^{n\alpha} Ł_\alpha \{u(x,y)\} - s^{(n-1)\alpha} u(0,y) - s^{(n-2)\alpha} u^{(\alpha)}(0,y) - \cdots - u^{((n-1)\alpha)}(0,y) \\ = Ł_\alpha \{f(x,y)\} - Ł_\alpha \left\{R_\alpha u(x,t)\right\}, \end{aligned} \tag{2.3}$$

or

$$\begin{aligned} Ł_\alpha \{u(x,y)\} &= \frac{1}{s^\alpha} u(0,y) + \frac{1}{s^{2\alpha}} u^{(\alpha)}(0,y) + \cdots + \frac{1}{s^{n\alpha}} u^{((n-1)\alpha)}(0,y) \\ &\quad + \frac{1}{s^{n\alpha}} Ł_\alpha \{f(x,y)\} - \frac{1}{s^{n\alpha}} Ł_\alpha \left\{R_\alpha u(x,y)\right\}. \end{aligned} \tag{2.4}$$

Operating with the Yang-Laplace inverse on both sides of (2.4) gives

$$u(x,y) = G(x,y) - Ł_\alpha^{-1} \left(\frac{1}{s^{n\alpha}} Ł_\alpha \left\{R_\alpha u(x,y)\right\} \right), \tag{2.5}$$

where $G(x,y)$ represents the term arising from the source term and the prescribed initial conditions. Now we apply the local fractional homotopy perturbation method:

$$u(x,y) = \sum_{n=0}^{\infty} p^{n\alpha} u_n(x,y), \tag{2.6}$$

where $p \in [0,1]$ is an embedding parameter.

Now, we substitute (2.6) in (2.5):

$$\sum_{n=0}^{\infty} p^{n\alpha} u_n(x,y) = G(x,y) - p^{\alpha} \left[Ł_{\alpha}^{-1} \left(\frac{1}{s^{n\alpha}} Ł_{\alpha} \left\{ R_{\alpha} \sum_{n=0}^{\infty} p^{n\alpha} u_n(x,y) \right\} \right) \right], \tag{2.7}$$

which is the coupling of the local fractional Laplace transform and homotopy perturbation method. Comparing the coefficients of like powers of p^{α}, the following approximations are obtained:

$$\begin{aligned} p^{0\alpha} &: u_0(x,y) = G(x,y), \\ p^{1\alpha} &: u_1(x,t) = -Ł_{\alpha}^{-1} \left\{ \frac{1}{s^{n\alpha}} Ł_{\alpha} \left\{ R_{\alpha} u_0(x,y) \right\} \right\}, \\ p^{2\alpha} &: u_2(x,t) = -Ł_{\alpha}^{-1} \left\{ \frac{1}{s^{n\alpha}} Ł_{\alpha} \left\{ R_{\alpha} u_1(x,y) \right\} \right\}, \\ &\vdots \end{aligned} \tag{2.8}$$

Proceeding in this same manner, the rest of the components $u_n(x,y)$ can be completely obtained and the series solution is thus entirely determined. Finally, we approximate the analytical solution $u(x,y)$ by truncated series:

$$u(x,y) = \lim_{N\to\infty} \sum_{n=0}^{N} u_n(x,y). \tag{2.9}$$

II. Local Fractional RDTM.

In the following the basic definitions and fundamental operations of the local fractional reduce differential transform method are shown [25].

Definition 1. If $u(x,y)$ is a local fractional analytical function in the domain of interest, then the local fractional spectrum function

$$U_k(y) = \frac{1}{\Gamma(1+k\alpha)} \left[\frac{\partial^{k\alpha} u(x,y)}{\partial x^{k\alpha}} \right]_{x=x_0}, \tag{2.10}$$

is reduce differential transformed of the function $u(x,y)$ via local fractional operator, where $k = 0, 1, 2, \ldots, n$ and $0 < \alpha \le 1$.

Definition 2. The inverse reduced differential transform of $U_k(y)$ via local fractional operator is defined as:

$$u(x,y) = \sum_{k=0}^{\infty} U_k(y)(x-x_0)^{k\alpha}. \tag{2.11}$$

From (2.1) and (2.2) we have

$$u(x,y) = \sum_{k=0}^{\infty} \frac{(x-x_0)^{k\alpha}}{\Gamma(1+k\alpha)} \left[\frac{\partial^{k\alpha} u(x,y)}{\partial x^{k\alpha}} \right]_{x=x_0}. \tag{2.12}$$

From (2.12), it is obvious that the local fractional reduce differential transform is derived from the local fractional Taylor theorems.

Whenever $x_0 = 0$, then (2.10) and (2.11) become

$$U_k(y) = \frac{1}{\Gamma(1+k\alpha)} \left[\frac{\partial^{k\alpha} u(x,y)}{\partial x^{k\alpha}} \right]_{x=0}, \tag{2.13}$$

$$u(x,y) = \sum_{k=0}^{\infty} U_k(y) x^{k\alpha}. \tag{2.14}$$

By using (2.10) and (2.11), the theorems of the local fractional transform method are deduced as follows:

Theorem 1. *If $\omega(x,y) = u(x,y) + v(x,y)$, then*

$$\Omega_k(y) = U_k(y) + V_k(y). \tag{2.15}$$

Theorem 2. *If $\omega(x,y) = au(x,y)$, then*

$$\Omega_k(y) = a\,U_k(y). \tag{2.16}$$

Theorem 3. *If $\omega(x,y) = u(x,y)v(x,y)$, then*

$$\Omega_k(y) = \sum_{l=0}^{k} U_l(y)\,V_{k-l}(y). \tag{2.17}$$

Theorem 4. *If $\omega(x,y) = \frac{\partial^{n\alpha}}{\partial x^{n\alpha}} u(x,y)$, then*

$$\Omega_k(y) = \frac{\Gamma\,(1+(k+n)\alpha)}{\Gamma(1+k\alpha)} U_{k+n}(y). \tag{2.18}$$

Theorem 5. *If $\omega(x,y) = \frac{x^{m\alpha}}{\Gamma(1+m\alpha)} \frac{y^{n\alpha}}{\Gamma(1+n\alpha)}$, then*

$$\Omega_k(y) = \frac{y^{n\alpha}}{\Gamma(1+n\alpha)} \frac{\delta_\alpha(k-m)}{\Gamma(1+m\alpha)}. \tag{2.19}$$

Theorem 6. *If $\omega(x,y) = \frac{\partial^{n\alpha} u(x,y)}{\partial y^{n\alpha}}$, then*

$$\Omega_k(y) = \frac{\partial^{n\alpha} U_k(y)}{\partial y^{n\alpha}}. \tag{2.20}$$

According to the local fractional RDTM, we can construct the following iteration for the equation (2.1) as:

$$\frac{\Gamma\,(1+(k+n)\alpha)}{\Gamma(1+k\alpha)} U_{k+n}(y) + R_\alpha \left[U_k(y) \right] = F_k(y), \tag{2.21}$$

where $U_k(y)$ and $F_k(y)$ are reduce differential transformed with local fractional operators of the functions $u(x,y)$ and $f(x,y)$ respectively.

3. Two Illustrative Examples

In this section we investigate the approximate solutions for the local fractional Poisson equations with different initial-boundary conditions.

Example 1. Let us consider the Poisson equation with LFDOs:

$$\frac{\partial^{2\alpha}u(x,y)}{\partial x^{2\alpha}}+\frac{\partial^{2\alpha}u(x,y)}{\partial y^{2\alpha}}=\frac{y^{\alpha}}{\Gamma(1+\alpha)}, \tag{3.1}$$

subject to the initial and boundary conditions

$$\begin{aligned} &u(x,0)=0, \quad u(x,l)=0 \\ &u(0,y)=\frac{y^{3\alpha}}{\Gamma(1+3\alpha)}, \\ &\frac{\partial^{\alpha}}{\partial x^{\alpha}}u(0,y)=\sin_{\alpha}(y^{\alpha}). \end{aligned} \tag{3.2}$$

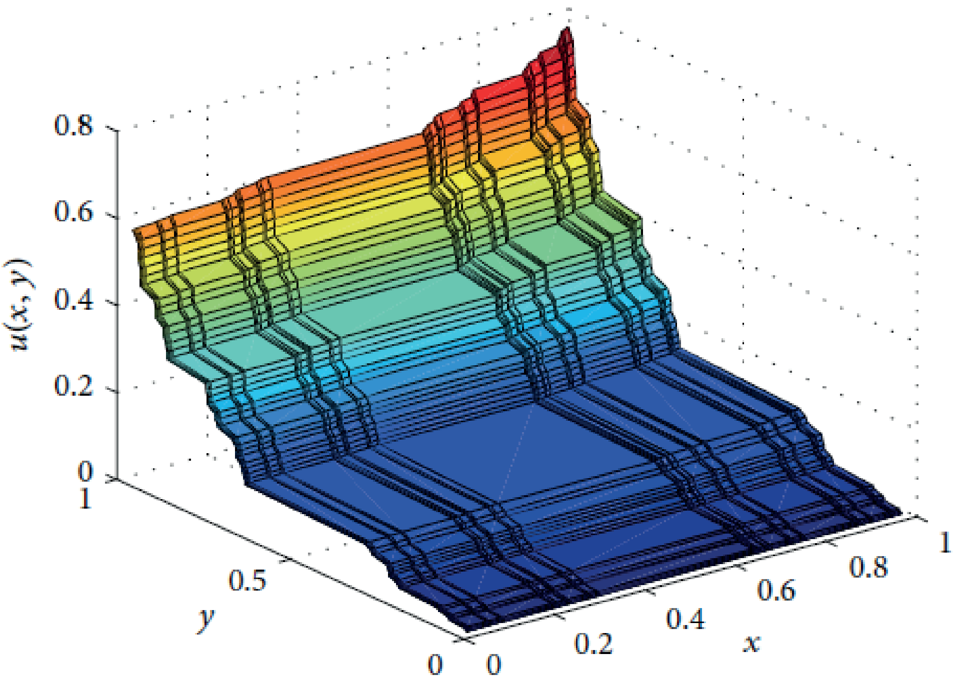

Figure 1: Plot of the approximate solution of (3.1) with the parameter $\alpha=\ln 2/\ln 3$.

I. Below we present the local fractional HPTM.

Applying the Yang-Laplace transform on both sides of (3.1), subject to the initial conditions (3.2), we have

$$Ł_{\alpha}\{u(x,y)\}=\frac{1}{s^{\alpha}}\frac{y^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{1}{s^{2\alpha}}\sin_{\alpha}(y^{\alpha})+\frac{1}{s^{3\alpha}}\frac{y^{\alpha}}{\Gamma(1+\alpha)}-\frac{1}{s^{2\alpha}}Ł_{\alpha}\left\{\frac{\partial^{2\alpha}u(x,y)}{\partial y^{2\alpha}}\right\}. \tag{3.3}$$

The inverse Yang-Laplace transform implies that

$$u(x,y)=\frac{y^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{x^{\alpha}}{\Gamma(1+\alpha)}\sin_{\alpha}(y^{\alpha})+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{y^{\alpha}}{\Gamma(1+\alpha)}-Ł_{\alpha}^{-1}\left(\frac{1}{s^{2\alpha}}Ł_{\alpha}\left\{\frac{\partial^{2\alpha}u(x,y)}{\partial y^{2\alpha}}\right\}\right). \tag{3.4}$$

Now applying the LFHPM, we get

$$\begin{aligned}\textstyle\sum_{n=0}^{\infty}p^{n\alpha}u_n(x,y) &= \frac{y^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{x^{\alpha}}{\Gamma(1+\alpha)}\sin_{\alpha}(y^{\alpha})+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{y^{\alpha}}{\Gamma(1+\alpha)}\\ &\quad -p^{\alpha}Ł_{\alpha}^{-1}\left(\frac{1}{s^{2\alpha}}Ł_{\alpha}\left\{\sum_{n=0}^{\infty}p^{n\alpha}\frac{\partial^{2\alpha}u_n(x,y)}{\partial y^{2\alpha}}\right\}\right).\end{aligned} \tag{3.5}$$

Comparing the coefficients of like powers of p^{α}, we have

$$\begin{aligned}p^{0\alpha}: u_0(x,y) &= \frac{y^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{x^{\alpha}}{\Gamma(1+\alpha)}\sin_{\alpha}(y^{\alpha})+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{y^{\alpha}}{\Gamma(1+\alpha)},\\ p^{1\alpha}: u_1(x,y) &= -Ł_{\alpha}^{-1}\left(\frac{1}{s^{2\alpha}}Ł_{\alpha}\left\{\frac{\partial^{2\alpha}u_0(x,y)}{\partial y^{2\alpha}}\right\}\right)\\ &= -\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{y^{\alpha}}{\Gamma(1+\alpha)}+\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}\sin_{\alpha}(y^{\alpha}),\\ p^{2\alpha}: u_2(x,y) &= -Ł_{\alpha}^{-1}\left(\frac{1}{s^{2\alpha}}Ł_{\alpha}\left\{\frac{\partial^{2\alpha}u_1(x,y)}{\partial y^{2\alpha}}\right\}\right)=\frac{x^{5\alpha}}{\Gamma(1+5\alpha)}\sin_{\alpha}(y^{\alpha}),\\ &\vdots\end{aligned}$$

Therefore, the series solution the equation (3.1) when $p\to 1$ will be as

$$\begin{aligned}u(x,y) &= \frac{y^{3\alpha}}{\Gamma(1+3\alpha)}+\sin_{\alpha}(y^{\alpha)}\left(\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{x^{3\alpha}}{\Gamma(1+3\alpha}+\frac{x^{5\alpha}}{\Gamma(1+5\alpha)}+\cdots\right)\\ &= \frac{y^{3\alpha}}{\Gamma(1+3\alpha)}+\sin_{\alpha}(y^{\alpha)}\sinh_{\alpha}(x^{\alpha}).\end{aligned} \tag{3.6}$$

II. As a next step we apply the local fractional RDTM.

To obtain solution of equation (3.1) using the local fractional RDTM, in view of equations (2.18) and (2.20), we can transform equation (3.1) to the following iteration

$$\frac{\Gamma(1+(k+2)\alpha)}{\Gamma(1+k\alpha)}U_{k+2}(y)+\frac{\partial^{2\alpha}U_k(y)}{\partial y^{2\alpha}}=\frac{y^{\alpha}}{\Gamma(1+\alpha)}\delta_{\alpha}(k), \tag{3.7}$$

or

$$U_{k+2}(y)=\frac{\Gamma(1+k\alpha)}{\Gamma(1+(k+2)\alpha)}\left(\frac{y^{\alpha}}{\Gamma(1+\alpha)}\delta_{\alpha}(k)-\frac{\partial^{2\alpha}U_k(y)}{\partial y^{2\alpha}}\right). \tag{3.8}$$

From the initial conditions (3.2), we obtain

$$U_0(y)=\frac{y^{3\alpha}}{\Gamma(1+3\alpha)},\quad U_1(y)=\frac{1}{\Gamma(1+\alpha)}\sin_{\alpha}(y^{\alpha}). \tag{3.9}$$

Now, substituting (3.9) into (3.8), we have the following $U_k(y)$ values successively:

$$
\begin{aligned}
U_2(y) &= \frac{1}{\Gamma(1+2\alpha)}\left(\frac{y^\alpha}{\Gamma(1+\alpha)}\delta_\alpha(0) - \frac{\partial^{2\alpha}U_0(y)}{\partial y^{2\alpha}}\right) \\
&= \left(\frac{y^\alpha}{\Gamma(1+\alpha)} - \frac{y^\alpha}{\Gamma(1+\alpha)}\right) = 0, \\
U_3(y) &= \frac{\Gamma(1+\alpha)}{\Gamma(1+3\alpha)}\left(\frac{y^\alpha}{\Gamma(1+\alpha)}\delta_\alpha(1) - \frac{\partial^{2\alpha}U_1(y)}{\partial y^{2\alpha}}\right) \\
&= \frac{\Gamma(1+\alpha)}{\Gamma(1+3\alpha)}\left(0 + \frac{1}{\Gamma(1+\alpha)}\sin_\alpha(y^\alpha)\right) \\
&= \frac{1}{\Gamma(1+3\alpha)}\sin_\alpha(y^\alpha), \\
U_4(y) &= \frac{\Gamma(1+2\alpha)}{\Gamma(1+4\alpha)}\left(\frac{y^\alpha}{\Gamma(1+\alpha)}\delta_\alpha(2) - \frac{\partial^{2\alpha}U_2(y)}{\partial y^{2\alpha}}\right) \\
&= 0, \\
U_5(y) &= \frac{\Gamma(1+3\alpha)}{\Gamma(1+5\alpha)}\left(\frac{y^\alpha}{\Gamma(1+\alpha)}\delta_\alpha(3) - \frac{\partial^{2\alpha}U_3(y)}{\partial y^{2\alpha}}\right) \\
&= \frac{\Gamma(1+3\alpha)}{\Gamma(1+5\alpha)}\left(0 + \frac{1}{\Gamma(1+3\alpha)}\sin_\alpha(y^\alpha)\right) \\
&= \frac{1}{\Gamma(1+5\alpha)}\sin_\alpha(y^\alpha), \\
&\vdots
\end{aligned}
$$

and so on.

Hence, the solution of (3.1) gives

$$
\begin{aligned}
u(x,y) &= \textstyle\sum_{k=0}^{\infty} U_k(y)\,x^{k\alpha} \\
&= \frac{y^{3\alpha}}{\Gamma(1+3\alpha)} + \sin_\alpha(y^{\alpha)}\left(\frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{x^{3\alpha}}{\Gamma(1+3\alpha} + \frac{x^{5\alpha}}{\Gamma(1+5\alpha)} + \cdots\right) \qquad (3.10) \\
&= \frac{y^{3\alpha}}{\Gamma(1+3\alpha)} + \sin_\alpha(y^{\alpha)}\sinh_\alpha(x^\alpha).
\end{aligned}
$$

From equations (3.6) and (3.10), approximate solution of the given problem equation (3.1) by using local fractional HPTM is the same results as that obtained by the local fractional RDTM and the local fractional variational iteration method [6].

Example 2. Consider the following Poisson equation with LFDOs:

$$
\frac{\partial^{2\alpha}u(x,y)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha}u(x,y)}{\partial y^{2\alpha}} = E_\alpha(y^\alpha), \qquad (3.11)
$$

subject to the initial and boundary conditions

$$
\begin{aligned}
&u(x,0)=0, \quad u(x,l)=0 \\
&u(0,y)=E_\alpha(y^\alpha), \\
&\frac{\partial^\alpha}{\partial x^\alpha}u(0,y)=\cos_\alpha(y^\alpha).
\end{aligned}
\tag{3.12}
$$

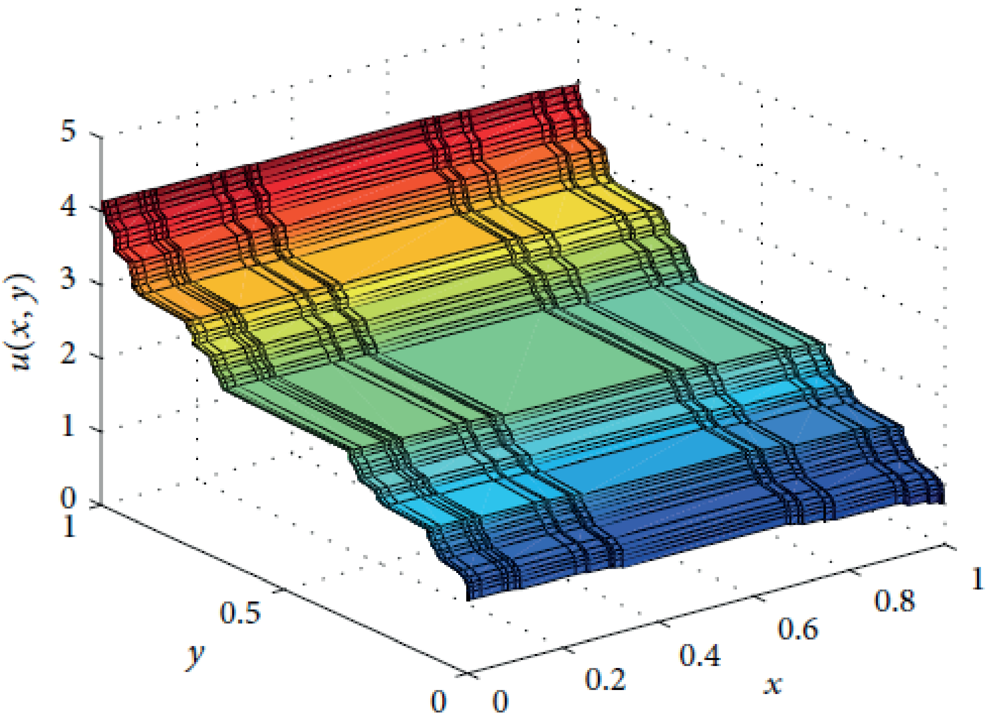

Figure 2: Plot of the nondifferentiable solution of (3.11) with the parameter $\alpha = \ln 2/\ln 3$.

Applying the Yang-Laplace transform on both sides of (3.11), we have

$$
\text{Ⱡ}_\alpha\left\{u(x,y)\right\}=\frac{1}{s^\alpha}E_\alpha(y^\alpha)+\frac{1}{s^{2\alpha}}\cos_\alpha(y^\alpha)+\frac{1}{s^{3\alpha}}E_\alpha(y^\alpha)-\frac{1}{s^{2\alpha}}\text{Ⱡ}_\alpha\left\{\frac{\partial^{2\alpha}u(x,y)}{\partial y^{2\alpha}}\right\}. \tag{3.13}
$$

The inverse Yang-Laplace transform implies that

$$
u(x,y)=E_\alpha(y^\alpha)+\frac{x^\alpha}{\Gamma(1+\alpha)}\cos_\alpha(y^\alpha)+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}E_\alpha(y^\alpha)-\text{Ⱡ}_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}\text{Ⱡ}_\alpha\left\{\frac{\partial^{2\alpha}u(x,y)}{\partial y^{2\alpha}}\right\}\right). \tag{3.14}
$$

Now applying the LFHPM, we get

$$
\begin{aligned}
\sum_{n=0}^{\infty}p^{n\alpha}u_n(x,y) \;=\; & E_\alpha(y^\alpha)+\frac{x^\alpha}{\Gamma(1+\alpha)}\cos_\alpha(y^\alpha)+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}E_\alpha(y^\alpha) \\
& -p^\alpha\text{Ⱡ}_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}\text{Ⱡ}_\alpha\left\{\sum_{n=0}^{\infty}p^{n\alpha}\frac{\partial^{2\alpha}u_n(x,y)}{\partial y^{2\alpha}}\right\}\right).
\end{aligned}
\tag{3.15}
$$

Comparing the coefficients of like powers of p^α, we have

$$\begin{aligned}
p^{0\alpha}\ :\ u_0(x,y) &= E_\alpha(y^\alpha) + \frac{x^\alpha}{\Gamma(1+\alpha)}\cos_\alpha(y^\alpha) + \frac{x^{2\alpha}}{\Gamma(1+2\alpha)}E_\alpha(y^\alpha) \\
p^{1\alpha}\ :\ u_1(x,y) &= -Ł_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}Ł_\alpha\left\{\frac{\partial^{2\alpha}u_0(x,y)}{\partial y^{2\alpha}}\right\}\right) \\
&= -\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}E_\alpha(y^\alpha) + \frac{x^{3\alpha}}{\Gamma(1+3\alpha)}\cos_\alpha(y^\alpha) - \frac{x^{4\alpha}}{\Gamma(1+4\alpha)}E_\alpha(y^\alpha) \\
p^{2\alpha}\ :\ u_1(x,y) &= -Ł_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}Ł_\alpha\left\{\frac{\partial^{2\alpha}u_1(x,y)}{\partial y^{2\alpha}}\right\}\right) \\
&= \frac{x^{4\alpha}}{\Gamma(1+4\alpha)}E_\alpha(y^\alpha) + \frac{x^{5\alpha}}{\Gamma(1+5\alpha)}\cos_\alpha(y^\alpha) + \frac{x^{6\alpha}}{\Gamma(1+6\alpha)}E_\alpha(y^\alpha) \\
&\ \vdots
\end{aligned}$$

Therefore, the series solution the equation (3.10) when $p \to 1$ will be as

$$\begin{aligned}
u(x,y) &= E_\alpha(y^\alpha) + \cos_\alpha(y^{\alpha)}\left(\frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{x^{3\alpha}}{\Gamma(1+3\alpha} + \frac{x^{5\alpha}}{\Gamma(1+5\alpha)}\cdots\right) \\
&= E_\alpha(y^\alpha) + \cos_\alpha(y^{\alpha)}\sinh_\alpha(x^\alpha).
\end{aligned} \tag{3.16}$$

(I) By using local fractional RDTM.

Taking the local fractional RDTM of (3.11), by using the basic operation in theorems, yields

$$\frac{\Gamma\left(1+(k+2)\alpha\right)}{\Gamma(1+k\alpha)}U_{k+2}(y) + \frac{\partial^{2\alpha}U_k(y)}{\partial y^{2\alpha}} = E_\alpha(y^\alpha)\delta_\alpha(k), \tag{3.17}$$

or

$$U_{k+2}(y) = \frac{\Gamma\left(1+k\alpha\right)}{\Gamma\left(1+(k+2)\alpha\right)}\left(E_\alpha(y^\alpha)\delta_\alpha(k) - \frac{\partial^{2\alpha}U_k(y)}{\partial\xi^{2\alpha}}\right). \tag{3.18}$$

From the initial condition (3.2), we obtain

$$U_0(y) = E_\alpha(y^\alpha), \quad U_1(y) = \frac{1}{\Gamma(1+\alpha)}\cos_\alpha(y^\alpha). \tag{3.19}$$

Now, substituting (3.19) into (3.18), we have the following $U_k(y)$ values successively

$$\begin{aligned}
U_2(y) &= \frac{1}{\Gamma(1+2\alpha)}\left(E_\alpha(y^\alpha)\delta_\alpha(0) - \frac{\partial^{2\alpha}U_0(y)}{\partial y^{2\alpha}}\right)\\
&= \left(E_\alpha(y^\alpha) - E_\alpha(y^\alpha)\right) = 0,\\
U_3(y) &= \frac{\Gamma(1+\alpha)}{\Gamma(1+3\alpha)}\left(\frac{y^\alpha}{\Gamma(1+\alpha)}\delta_\alpha(1) - \frac{\partial^{2\alpha}U_1(y)}{\partial y^{2\alpha}}\right)\\
&= \frac{\Gamma(1+\alpha)}{\Gamma(1+3\alpha)}\left(0 + \frac{1}{\Gamma(1+\alpha)}\sin_\alpha(y^\alpha)\right)\\
&= \frac{1}{\Gamma(1+3\alpha)}\sin_\alpha(y^\alpha),\\
U_4(y) &= \frac{\Gamma(1+2\alpha)}{\Gamma(1+4\alpha)}\left(\frac{y^\alpha}{\Gamma(1+\alpha)}\delta_\alpha(2) - \frac{\partial^{2\alpha}U_2(y)}{\partial y^{2\alpha}}\right)\\
&= 0,\\
U_5(y) &= \frac{\Gamma(1+3\alpha)}{\Gamma(1+5\alpha)}\left(\frac{y^\alpha}{\Gamma(1+\alpha)}\delta_\alpha(3) - \frac{\partial^{2\alpha}U_3(y)}{\partial y^{2\alpha}}\right)\\
&= \frac{\Gamma(1+3\alpha)}{\Gamma(1+5\alpha)}\left(0 + \frac{1}{\Gamma(1+3\alpha)}\sin_\alpha(y^\alpha)\right)\\
&= \frac{1}{\Gamma(1+5\alpha)}\sin_\alpha(y^\alpha),\\
&\vdots
\end{aligned}$$

and so on.

Hence, the solution of (3.11) gives

$$\begin{aligned}
u(x,y) &= \sum_{k=0}^{\infty} U_k(y)\, x^{k\alpha}\\
&= \frac{y^{3\alpha}}{\Gamma(1+3\alpha)} + \sin_\alpha(y^{\alpha)}\left(\frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{x^{3\alpha}}{\Gamma(1+3\alpha} + \frac{x^{5\alpha}}{\Gamma(1+5\alpha)} + \cdots\right)\\
&= \frac{y^{3\alpha}}{\Gamma(1+3\alpha)} + \sin_\alpha(y^{\alpha)}\sinh_\alpha(x^\alpha).
\end{aligned} \tag{3.20}$$

From equations (3.16), (3.20) and (3.10), approximate solution of the given problem equation (3.11) by using local fractional HPTM is the same results as that obtained by the local fractional RDTM and the local fractional variational iteration method [6].

4. Conclusions

In this work, the reduced differential transform method (RDTM) and homotopy perturbation transform method (HPTM), have been successfully applied for the Poisson equation within local fractional derivative operators. It can be concluded that, RDTM is a very powerful and efficient technique for finding approximate solutions for wide classes of problems and can be applied to many complicated linear and non-linear problems, and does not require linearization, discretization or perturbation. There are two important points to make here. First, the local fractional RDTM and the local fractional HPTM provide the solutions in terms of convergent series with easily computable components. Second, it seems that the approximate solution in examples using RDTM converges faster than the approximate solution using HPTM. Our goal in the future is to apply the RDTM to nonlinear PDEs that arises in mathematical physics.

Competing Interests

The author declares no competing interests.

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