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Research Article

Sumudu Decomposition Method for Solving Fractional Delay Differential Equations

Hassan Eltayeb[1] and Eltayeb Abdeldaim[2]

[1] *Mathematics Department College of Science, King Saud University*
[2] *Omdurman Islamic University College of Science and technology, Sudan*

Abstract. In this paper, The Sumudu transform decomposition method is applied to solve the linear and nonlinear fractional delay differential equations (DDEs). Numerical examples are presented to support our method.

Keywords: Sumudu transform decomposition, linear and nonlinear fractional delay differential equations

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Corresponding Author
Hassan Eltayeb
hgadain@ksu.edu.sa

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1. Introduction

In the literature there are a kind of integral transforms used in physics and engineering, the integral transforms were extensively used to solve the differential equations, several works on the theory and application of integral transforms such as Laplace, Fourier, Mellin and Hankel.

Watugala [1] introduce a new integral transform named the Sumudu transform and applied it to solution of ordinary differential equation in control engineering problems for properties of Sumudu transform see [2], [3], [4] and [5]. In [18] Maria Ragusa proved a sufficient condition for commutators of fractional integral operators. The Sumudu transform is defined over the set of the functions:

$$A = \left\{ f(t) : \exists M, \tau_1, \tau_2 > 0, f(t) < Me^{\frac{t}{\tau_j}}, \text{ if } t \in (-1)^j \times [0, \infty) \right\}.$$

By the following formula:

$$F(u) = S\left[f(t)\right] = \frac{1}{u} \int_0^\infty e^{\frac{-t}{u}} f(t)dt, \quad u \in (-\tau_1, \tau_2).$$

Delay differential equations arise when the rate of change of a time dependent process in its mathematical modeling is not only determined by its present state but also at certain past estate known as its history. Introduction of delays in models enriches the dynamics of such models and allow a precise description of real life phenomena. DDEs arise frequently in single processing, digital images, control system [8], lasers, traffic models [6], metal cutting, population dynamic [9], chemical kinetics [7], and in many physical phenomena.

Theorem 1. *If $F^n(u)$ is the Sumudu transform of n-th order derivative of $f^n(t)$ then for $n \geq 1$,*

$$F^n(u) = \frac{F(u)}{u^n} - \sum_{k=0}^{n-1} \frac{f^k(0)}{u^{n-k}}.$$

For more details see [4].

Analysis of the Method

In this paper we will consider a class of nonlinear delay differential equation of the form:

$$\frac{d^n y}{dt^n} + R(y) + N(t-\tau) = f(t), \tag{1}$$

with the initial condition:

$$u^k(0) = u_0^k, \tag{2}$$

where $y = y(t)$, R is a linear bounded operator and $f(t)$ is a given continuous function N is a nonlinear bounded operator and $\frac{d^n y}{dt^n}$ is the term of the highest order derivative.

The Sumudu decomposition method consists of applying the Sumudu transform first on both side of (1) to give:

$$S\left[\frac{d^n y}{dt^n}\right] + S\left[R(y)\right] + S\left[N(t-\tau)\right] = S\left[f(t)\right].$$

By Theorem 1, we have

$$\frac{S\left(y(t)\right)}{u^n} - \frac{C}{u^{n-k}} + S\left[R(y)\right] + S\left[N(t-\tau)\right] = S\left[f(t)\right],$$

where $C = \sum_{k=0}^{n-1} f^k(0)$,

$$S\left(y(t)\right) = u^k C - u^n S\left[R(y)\right] - u^n S\left[N(t-\tau)\right] + u^n S\left[f(t)\right]. \tag{3}$$

The standard Sumudu decomposition method defines the solution $y(t)$ by the series:

$$y(t) = \sum_{n=0}^{\infty} y_n(t), \tag{4}$$

the nonlinear operator is decomposed as:

$$N(t-\tau) = \sum_{n=0}^{\infty} A_n, \tag{5}$$

where A_n is the a domain polynomial of $y_0, y_1, y_2, \ldots, y_n$ that are given by:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N(\sum_{n=0}^{\infty} \lambda^n y_n)\right]_{\lambda=0}, \quad n = 0, 1, 2, \ldots \tag{6}$$

The first a domain polynomials are given by:

$$\begin{aligned} A_0 &= f(y_0) \\ A_1 &= y_1 f^1(y_0) \\ A_2 &= y_2 f^1(y_0) + \tfrac{1}{2!} y_1^2 f^2(y_0) \\ A_3 &= y_3 f^1(y_0) + y_1 y_2 f^2(y_0) + \tfrac{1}{3!} y_1^3 f^3(y_0). \end{aligned} \tag{7}$$

Apply (4) and (5) into (3) we have:

$$S\left[\sum_{n=0}^{\infty} y_n\right] = u^k C - u^n S\left[R\sum_{n=0}^{\infty} y_n\right] - u^n S\left[\sum_{n=0}^{\infty} A_n\right] + u^n S\left[f(t)\right], \tag{8}$$

comparing both side of (8):

$$S\left[y_0\right] = u^k C + u^n S\left[f(t)\right] \tag{9}$$

$$S\left[y_1\right] = -u^n S\left[R y_0\right] - u^n S\left[A_0\right] \tag{10}$$

$$S\left[y_2\right] = -u^n S\left[R y_1\right] - u^n S\left[A_1\right]. \tag{11}$$

In general the recursive relation is given by:

$$S\left[y_n\right] = -u^n S\left[R y_{n-1}\right] - u^n S\left[A_{n-1}\right], \quad n \geq 1 \tag{12}$$

applying inverse Sumudu transform to (9)–(12) then:

$$y_0 = H(t) \tag{13}$$

$$y_n = -S^{-1}\left[u^n S\left[R y_{n-1}\right] + u^n S\left[A_{n-1}\right]\right], \quad n \geq 1 \tag{14}$$

Where $H(t)$ is a function that a rises from the source term and prescribed initial conditions.

Numerical Examples

Example 1. Consider the nonlinear delay differential equation of first order:

$$y'(t) = 1 - 2y^2\left(\frac{t}{2}\right), \quad 0 \leq t \leq 1, \ y(0) = 0. \tag{15}$$

Apply Sumudu transform to both side of equation (15):

$$S\left[y'(t)\right] = S\left[1 - 2y^2\left(\frac{t}{2}\right)\right].$$

Using Theorem 1 and initial condition we have:

$$\frac{Y(u) - y(0)}{u} = 1 - S\left[2y^2\left(\frac{t}{2}\right)\right],$$

$$\frac{Y(u)}{u} = 1 - S\left[2y^2\left(\frac{t}{2}\right)\right],$$

$$S\left[y(t)\right] = u - uS\left[2y^2\left(\frac{t}{2}\right)\right]. \tag{16}$$

Applying the inverse Sumudu transform to (16) we have:

$$y(t) = S^{-1}\left[u\right] - S^{-1}\left[uS\left(2y^2\left(\frac{t}{2}\right)\right)\right],$$

$$y_0(t) = S^{-1}\left[u\right] = t,$$

$$y_0\left(\frac{t}{2}\right) = \frac{t}{2}, \tag{17}$$

$$y_{n+1}(t) = -S^{-1}\left[uS\left(2A_n\right)\right]. \tag{18}$$

From equation (7)

$$\begin{aligned} A_0 &= y_0^2\left(\tfrac{t}{2}\right) \\ A_1 &= 2y_0\left(\tfrac{t}{2}\right)y_1\left(\tfrac{t}{2}\right) \\ A_2 &= 2y_2\left(\tfrac{t}{2}\right)y_0\left(\tfrac{t}{2}\right) + y_1^2\left(\tfrac{t}{2}\right), \end{aligned} \tag{19}$$

At $n = 0$ in equation (18):

$$y_1(t) = -S^{-1}\left[uS\left(2A_0\right)\right], \tag{20}$$

substituting equation (19) in (20) we get:

$$y_1(t) = -S^{-1}\left[uS\left(2y_0^2\left(\frac{t}{2}\right)\right)\right],$$

$$y_1(t) = -S^{-1}\left[uS\left(2(\tfrac{t}{4})\right)\right] = -S^{-1}\left[uS\left(\frac{t^2}{2}\right)\right],$$

$$y_1(t) = -S^{-1}\left[u\left(u^2\right)\right] = -S^{-1}\left[u^3\right] = -\frac{t^3}{3!},$$

$$y_1\left(\frac{t}{2}\right) = -\frac{\left(\frac{t}{2}\right)^3}{3!} = -\frac{t^3}{48}. \tag{21}$$

At $n = 1$ in equation (18) we have:

$$y_2(t) = -S^{-1}\left[uS\left(2A_1\right)\right], \tag{22}$$

substituting equations (19) into (22):

$$y_2(t) = -S^{-1}\left[uS\left(2(2y_0\left(\frac{t}{2}\right)y_1\left(\frac{t}{2}\right)\right)\right],$$

$$y_2(t) = -S^{-1}\left[uS\left(\frac{4t}{2}\left(\frac{-t^3}{48}\right)\right)\right] = S^{-1}\left[uS\left(\frac{t^4}{24}\right)\right],$$

$$y_2(t) = S^{-1}\left[uS\left(\frac{4!\,u^4}{24}\right)\right] = S^{-1}\left[u^5\right] = \frac{t^5}{5!} = \frac{t^5}{120},$$

$$y_2\left(\frac{t}{2}\right) = \frac{t^5}{3840}, \tag{23}$$

At $n = 2$ in equation (18) we have:

$$y_3(t) = -S^{-1}\left[uS\left(2A_2\right)\right] \tag{24}$$

Substituting equations (19) and (24):

$$y_3(t) = -S^{-1}\left[uS\left(2(2y_2\left(\frac{t}{2}\right)y_0\left(\frac{t}{2}\right) + y_1^2\left(\frac{t}{2}\right)\right)\right],$$

$$y_3(t) = -S^{-1}\left[2uS\left(\frac{2t^5}{3840}\left(\frac{t}{2}\right) + \left(\frac{-t^3}{48}\right)^2\right)\right],$$

$$y_3(t) = -S^{-1}\left[2uS\left(\frac{t^6}{3840} + \frac{t^6}{2304}\right)\right],$$

$$y_3(t) = -S^{-1}\left[2u\left(\frac{u^6.6!}{3840} + \frac{u^6.6!}{2304}\right)\right],$$

$$y_3(t) = -S^{-1}\left[u^7\right] = -\frac{t^7}{7!} = -\frac{t^7}{5040}.$$

The series solution is given by:

$$y(t) = y_0(t) + y_1(t) + y_2(t) + \cdots,$$

$$y(t) = t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040} + \cdots.$$

The exact solution is

$$y(t) = \sin(t)$$

Example 2. Consider the linear delay differential equation of first order

$$y'(t) - y\left(\frac{t}{2}\right) = 0, \quad 0 < \alpha \leq 1,\ 0 < t \leq 1, \tag{25}$$

with initial condition $y(0) = 1$, apply Sumudu transform to both side of (25)

$$S\left[y'(t)\right] = S\left[y\left(\frac{t}{2}\right)\right],$$

using Theorem 1 and initial condition:

$$\frac{Y(u) - y(0)}{u} = S\left[y\left(\frac{t}{2}\right)\right],$$

$$\frac{Y(u) - 1}{u} = S\left[y\left(\frac{t}{2}\right)\right],$$

$$Y(u) = 1 + uS\left[y\left(\frac{t}{2}\right)\right],$$

$$S\left[y(t)\right] = 1 + uS\left[y\left(\frac{t}{2}\right)\right]. \tag{26}$$

Applying the inverse Sumudu transform to (26):

$$y(t) = S^{-1}\left[1\right] + S^{-1}\left[uS\left[y\left(\frac{t}{2}\right)\right]\right],$$

$$y_0(t) = S^{-1}\left[1\right] = 1,$$

$$y_0\left(\frac{t}{2}\right) = 1, \tag{27}$$

$$y_{n+1}(t) = S^{-1}\left[uS\left[y_n\left(\frac{t}{2}\right)\right]\right], \tag{28}$$

at $n = 0$ in equation (28):

$$y_1(t) = S^{-1}\left[uS\left[y_0\left(\frac{t}{2}\right)\right]\right],$$

$$y_1(t) = S^{-1}\left[uS\left[1\right]\right],$$

$$y_1(t) = S^{-1}\left[u\right] = t,$$

$$y_1\left(\frac{t}{2}\right) = \frac{t}{2}, \tag{29}$$

at $n = 1$ in equation (28):

$$y_2(t) = S^{-1}\left[uS\left[y_1\left(\frac{t}{2}\right)\right]\right],$$

$$y_2(t) = S^{-1}\left[uS\left[\frac{t}{2}\right]\right],$$

$$y_2(t) = S^{-1}\left[\frac{u^2}{2}\right] = \frac{t^2}{4},$$

at $n = 1$ in equation (28):

$$y_3(t) = S^{-1}\left[uS\left[y_2\left(\frac{t}{2}\right)\right]\right] = S^{-1}\left[uS\left[\frac{t^2}{16}\right]\right] = S^{-1}\left[\frac{u^3}{8}\right] = \frac{t^3}{48}.$$

The series solution is given by:

$$y(t) = y_0(t) + y_1(t) + y_2(t) + \cdots,$$

$$y(t) = 1 + t + \frac{t^2}{4} + \frac{t^3}{48} + \cdots$$

The exact solution is $y(t) = \sum_{k=0}^{\infty} \frac{(\frac{1}{2})^{\frac{1}{2}k(k-1)}}{k!} t^k$.

Fractional Delay Differential Equation

In this section we apply the Sumudu decomposition method to solve linear and nonlinear fractional delay differential equation.

Definition 1. The Sumudu transform of the Caputo fractional derivative is defined as follows:

$$S\left[D^\alpha f(t)\right] = u^{-\alpha} S\left[f(t)\right] - \sum_{k=0}^{n-1} u^{-\alpha+k} f^k(0), \quad n-1 < \alpha \leq n,$$

for more details see [16].

2. Analysis of the Method of Fractional Order

Here we will consider a class of nonlinear delay differential equation of the form:

$$D^\alpha y(t) + R(y) + N(t-\tau) = f(t), \quad \tau \in R, \; t < \tau, \; n-1 < \alpha \leq n, \tag{30}$$

with the initial condition:

$$u^k(0) = u_0^k, \tag{31}$$

where R is a linear bounded operator and $f(t)$ is a given continuous function N is a nonlinear bounded operator and $D^\alpha y(t)$ is the term of the fractional order derivative.

The Sumudu decomposition method consists of applying the Sumudu transform first on both side of (30) to give:

$$S\left[D^\alpha y(t)\right] + S\left[R(y)\right] + S\left[N(t-\tau)\right] = S\left[f(t)\right],$$

by Definition 1,

$$\frac{S\left(y(t)\right)}{u^\alpha} - \frac{C}{u^{\alpha-k}} + S\left[R(y)\right] + S\left[N(t-\tau)\right] = S\left[f(t)\right].$$

Where $C = \sum_{k=0}^{n-1} f^k(0)$

$$S\left(y(t)\right) = u^k C + u^\alpha S\left[f(t)\right] - u^\alpha S\left[R(y)\right] - u^\alpha S\left[N(t-\tau)\right]. \tag{32}$$

The standard Sumudu decomposition method define the solution $y(t)$ by the series:

$$y(t) = \sum_{n=0}^{\infty} y_n(t), \tag{33}$$

the nonlinear operator is decomposed as:

$$N(t-\tau) = \sum_{n=0}^{\infty} A_n \tag{34}$$

Where A_n as in (6). The first a domain polynomials are given as in (7). Apply (33) and (34) in (32) we have:

$$S\left[\sum_{n=0}^{\infty} y_n\right] = u^k C + u^\alpha S\left[f(t)\right] - u^\alpha S\left[R\sum_{n=0}^{\infty} y_n\right] - u^\alpha S\left[\sum_{n=0}^{\infty} A_n\right] \tag{35}$$

Comparing both side of (35):

$$S\left[y_0\right] = u^k C + u^\alpha S\left[f(t)\right], \tag{36}$$

$$S\left[y_1\right] = -u^\alpha S\left[Ry_0\right] - u^\alpha S\left[A_0\right], \tag{37}$$

$$S\left[y_2\right] = -u^\alpha S\left[Ry_1\right] - u^\alpha S\left[A_1\right]. \tag{38}$$

In general the recursive relation is given by:

$$S\left[y_n\right] = -u^\alpha S\left[Ry_{n-1}\right] - u^\alpha S\left[A_{n-1}\right], \quad n \geq 1, \tag{39}$$

applying inverse Sumudu transform to (36)–(39) then:

$$y_0 = H(t), \tag{40}$$

$$y_n = -S^{-1}\left[u^\alpha S\left[Ry_{n-1}\right] + u^\alpha S\left[A_{n-1}\right]\right], \quad n \geq 1, \tag{41}$$

where $H(t)$ is a function that a rises from the source term and prescribed initial conditions.

Example 3. Consider the nonlinear delay differential equation of first order:

$$D^\alpha y(t) = 1 - 2y^2\left(\frac{t}{2}\right), \quad 0 \leq t \leq 1,\ 0 < \alpha \leq 1, \tag{42}$$

$$y(0) = 0, \tag{43}$$

apply Sumudu transform to both side of equation (42):

$$S\left[D^\alpha y(t)\right] = S\left[1 - 2y^2\left(\frac{t}{2}\right)\right],$$

by using Definition 1 and initial condition (43) we have:

$$\frac{Y(u)-y(0)}{u^{\alpha}}=1-S\left[2y^{2}\left(\frac{t}{2}\right)\right],$$

$$\frac{Y(u)}{u^{\alpha}}=1-S\left[2y^{2}\left(\frac{t}{2}\right)\right],$$

$$S\left[y(t)\right]=u^{\alpha}-u^{\alpha}S\left[2y^{2}\left(\frac{t}{2}\right)\right]. \tag{44}$$

Applying the inverse Sumudu transform to (44) we have:

$$y(t)=S^{-1}\left[u^{\alpha}\right]-S^{-1}\left[u^{\alpha}S\left(2y^{2}\left(\frac{t}{2}\right)\right)\right],$$

$$y_0(t)=S^{-1}\left[u^{\alpha}\right]=\frac{t^{\alpha}}{\Gamma(\alpha+1)},$$

$$y_0\left(\frac{t}{2}\right)=\frac{\left(\frac{t}{2}\right)^{\alpha}}{\Gamma(\alpha+1)}=\frac{t^{\alpha}}{2^{\alpha}\Gamma(\alpha+1)}, \tag{45}$$

$$y_{n+1}(t)=-S^{-1}\left[u^{\alpha}S\left(2A_n\right)\right], \tag{46}$$

From equation (7), we have

$$\begin{aligned}
A_0 &= y_0^2\left(\tfrac{t}{2}\right)\\
A_1 &= 2y_0\left(\tfrac{t}{2}\right)y_1\left(\tfrac{t}{2}\right)\\
A_2 &= 2y_2\left(\tfrac{t}{2}\right)y_0\left(\tfrac{t}{2}\right)+y_1^2\left(\tfrac{t}{2}\right),
\end{aligned} \tag{47}$$

at $n=0$ in equation (46):

$$y_1(t)=-S^{-1}\left[u^{\alpha}S\left(2A_0\right)\right], \tag{48}$$

substituting equation (47) in (48) we get:

$$y_1(t)=-S^{-1}\left[u^{\alpha}S\left(2y_0^2\left(\frac{t}{2}\right)\right)\right],$$

$$y_1(t)=-S^{-1}\left[u^{\alpha}S\left(2\left(\frac{t^{\alpha}}{2^{\alpha}\Gamma(\alpha+1)}\right)^{2}\right)\right]=-S^{-1}\left[u^{\alpha}S\left(\frac{t^{2\alpha}}{2^{2\alpha-1}(\Gamma(\alpha+1))^{2}}\right)\right],$$

$$y_1(t)=-S^{-1}\left[u^{\alpha}\left(\frac{u^{2\alpha}\Gamma(2\alpha+1)}{2^{2\alpha-1}(\Gamma(\alpha+1))^{2}}\right)\right],$$

$$y_1(t)=-S^{-1}\left[\frac{u^{3\alpha}\Gamma(2\alpha+1)}{2^{2\alpha-1}(\Gamma(\alpha+1))^{2}}\right],$$

$$y_1(t) = -\frac{t^{3\alpha}\Gamma(2\alpha+1)}{2^{2\alpha-1}(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)},$$

$y_1(t) = -A\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$, where $A = \frac{\Gamma(2\alpha+1)}{2^{2\alpha-1}(\Gamma(\alpha+1))^2}$,

$$y_1\left(\frac{t}{2}\right) = -A\frac{\left(\frac{t}{2}\right)^{3\alpha}}{\Gamma(3\alpha+1)} = -A\frac{t^{3\alpha}}{2^{3\alpha}\Gamma(3\alpha+1)}, \tag{49}$$

at $n = 1$ in equation (46) we have:

$$y_2(t) = -S^{-1}\left[u^\alpha S\left(2A_1\right)\right], \tag{50}$$

substituting equations (47) in (50):

$$y_2(t) = -S^{-1}\left[u^\alpha S\left(2(2y_0\left(\frac{t}{2}\right)y_1\left(\frac{t}{2}\right)\right)\right],$$

$$y_2(t) = -S^{-1}\left[u^\alpha S\left(4y_0\left(\frac{t}{2}\right)y_1\left(\frac{t}{2}\right)\right)\right],$$

$$y_2(t) = -S^{-1}\left[u^\alpha S\left(4\left(\frac{t^\alpha}{2^\alpha\Gamma(\alpha+1)}\right)\left(-A\frac{t^{3\alpha}}{2^{3\alpha}\Gamma(3\alpha+1)}\right)\right)\right],$$

$$y_2(t) = -S^{-1}\left[u^\alpha S\left(-A\frac{t^{4\alpha}}{2^{4\alpha-2}\Gamma(3\alpha+1)}\right)\right],$$

$$y_2(t) = -S^{-1}\left[u^\alpha\left(-A\frac{u^{4\alpha}\Gamma(4\alpha+1)}{2^{4\alpha-2}\Gamma(3\alpha+1)}\right)\right],$$

$$y_2(t) = -S^{-1}\left[-A\frac{u^{5\alpha}\Gamma(4\alpha+1)}{2^{4\alpha-2}\Gamma(3\alpha+1)}\right],$$

$$y_2(t) = A\frac{t^{5\alpha}\Gamma(4\alpha+1)}{2^{4\alpha-2}\Gamma(3\alpha+1)\Gamma(5\alpha+1)}.$$

The series solution is given by:

$$y(t) = y_0(t) + y_1(t) + y_2(t) + \cdots$$

$$y(t) = \frac{t^\alpha}{\Gamma(\alpha+1)} - A\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + A\frac{t^{5\alpha}\Gamma(4\alpha+1)}{2^{4\alpha-2}\Gamma(3\alpha+1)\Gamma(5\alpha+1)} + \cdots.$$

In particular case$\alpha = 1$then we have:

$$y(t) = \frac{t}{\Gamma(2)} - \frac{t^3}{\Gamma(4)} + \frac{t^5\Gamma(5)}{2^2\Gamma(4)\Gamma(6)} + \cdots,$$

$$y(t) = t - \frac{t^3}{6} + \frac{t^5}{120} + \cdots.$$

The exact solution when $\alpha = 1$ is given by $y(t) = \sin(t)$

Example 4. Consider the nonlinear delay differential equation of first order

$$D^{\alpha} y(t) - y\left(\frac{t}{2}\right) = 0, \quad 0 < \alpha \leq 1,\ 0 < t \leq 1, \tag{51}$$

with initial condition $y(0) = 1$.

Apply Sumudu transform to both side of (51)

$$S\left[D^{\alpha} y(t)\right] = S\left[y\left(\frac{t}{2}\right)\right],$$

Using Definition 1 and initial condition:

$$\frac{Y(u) - y(0)}{u^{\alpha}} = S\left[y\left(\frac{t}{2}\right)\right],$$

$$\frac{Y(u) - 1}{u^{\alpha}} = S\left[y\left(\frac{t}{2}\right)\right],$$

$$Y(u) = 1 + u^{\alpha} S\left[y\left(\frac{t}{2}\right)\right],$$

$$S\left[y(t)\right] = 1 + u^{\alpha} S\left[y\left(\frac{t}{2}\right)\right]. \tag{52}$$

Applying the inverse Sumudu transform to (52):

$$y(t) = S^{-1}\left[1\right] + S^{-1}\left[u^{\alpha} S\left[y\left(\frac{t}{2}\right)\right]\right],$$

$$y_0(t) = S^{-1}\left[1\right] = 1,$$

$$y_0\left(\frac{t}{2}\right) = 1, \tag{53}$$

$$y_{n+1}(t) = S^{-1}\left[u^{\alpha} S\left[y_n\left(\frac{t}{2}\right)\right]\right], \tag{54}$$

at $n = 0$ in equation (54):

$$y_1(t) = S^{-1}\left[u^{\alpha} S\left[y_0\left(\frac{t}{2}\right)\right]\right],$$

$$y_1(t) = S^{-1}\left[u^{\alpha} S\left[1\right]\right],$$

$$y_1(t) = S^{-1}\left[u^{\alpha}\right] = \frac{t^{\alpha}}{\Gamma(\alpha+1)},$$

$$y_1\left(\frac{t}{2}\right) = \frac{t^{\alpha}}{2^{\alpha}\Gamma(\alpha+1)}, \tag{55}$$

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at $n = 1$ in equation (54):

$$y_2(t) = S^{-1}\left[u^\alpha S\left[y_1\left(\frac{t}{2}\right)\right]\right],$$

$$y_2(t) = S^{-1}\left[u^\alpha S\left[\frac{t^\alpha}{2^\alpha\Gamma(\alpha+1)}\right]\right],$$

$$y_2(t) = S^{-1}\left[\frac{u^{2\alpha}\Gamma(\alpha+1)}{2^\alpha}\right] = \frac{t^{2\alpha}\Gamma(\alpha+1)}{2^\alpha\Gamma(2\alpha+1)}.$$

The series solution is given by:

$$y(t) = y_0(t) + y_1(t) + y_2(t) + \cdots,$$

$$y(t) = 1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}\Gamma(\alpha+1)}{2^\alpha\Gamma(2\alpha+1)} + \cdots.$$

In particular case$\alpha = 1$then we have:

$$y(t) = 1 + t + \frac{t^2}{4} + \cdots$$

The exact solution is given by $y(t) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)^{\frac{1}{2}k(k-1)}}{k!} t^k$.

Conclusion

In this paper the Sumudu decomposition method has been successfully applied to solve delay and fractional delay differential equations. The method is very powerful and efficient in finding the exact solution.

Competing Interests

The authors declare that they have no competing interests.

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