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Research Article

Uniqueness and Differential Polynomials of Meromorphic Functions

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Abstract. Using Nevanlinna value distribution theory, we study the uniqueness of meromorphic functions concerning differential polynomials and prove a theorem. The results of the paper improve the recent results due to Waghamore and Anand [5].

Keywords: Meromorphic functions, Uniqueness, Differential Polynomial, Sharing Values.

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1. Introduction and Definitions

In this paper we use the standard definitions and notations of the value distribution theory [3]. Let f and g be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. For $a \in \mathbb{C} \cup \{\infty\}$ we say that f and g share the value a CM (counting multiplicities) if the a-points of f and g coincide in locations and multiplicities. If we do not consider the multiplicities, we say that f and g share the value a IM(ignoring multiplicities).

We denote by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ possibly outside a set of finite linear measure. A meromorphic function $a = a(z)$ is called small function of f if $T(r, a) = S(r, f)$.

Definition 1.1. Let f be a nonconstant meromorphic function and $a \in \mathbb{C} \cup \{\infty\}$. we denote by $N(r, a; f)$ the counting function of all the a-points of f and by $\overline{N}(r, a; f)$ the corresponding one for which the multiplicity is not counted.

Definition 1.2. Let f be a nonconstant meromorphic function and $a \in \mathbb{C} \cup \{\infty\}$. For a positive integer k we denote by $N_k(r, a; f)$ the counting function of those a-points of f, where an a-point of multiplicity m is counted m times if and only if $m \le k$ and k times if and only if $m > k$.

In 1996 Fang and Hua [2] proved the following theorem:

Theorem A (see [2]). *Let f and g be two nonconstant entire functions. Also let $n \ge 6$ be a positive integer. If $f^n f'$ and $g^n g'$ share the value 1 CM, then one of the following holds*

(i) $f(z) = c_1 e^{cz}, g(z) = c_2 e^{-cz}$, *where c_1, c_2 and c are three constants satisfying* $(c_1 c_2)^{n+1} c^2 = -1$.

(ii) $f = kg$ *for a constant k such that* $k^{n+1} = 1$.

In 2004, Lin and Yi [4] proved the following theorem:

Theorem B (see [4]). *Let f and g be two transcendental meromorphic functions and let $n \ge 13$ be an integer. If $f^n(f-1)^2 f'$ and $g^n(g-1)^2 g'$ share z CM, then $f(z) \equiv g(z)$.*

Recently H. P. Waghamore and S. Anand [5] proved the following theorem:

Theorem C (see [5]). *Let f and g be two nonconstant meromorphic functions and n, m be positive integers such that $n \geq m + 10$. If $f^n(f-1)^m f'$ and $g^n(g-1)^m g'$ share z CM, f and g share the value ∞ IM, then $f(z) \equiv g(z)$.*

Since

$$f^n f' = \frac{1}{n+1}(f^{n+1})'$$

and

$$\begin{aligned} f^n(f-1)^m f' &= \tfrac{1}{n+1}(f^{n+1})'(f-1)^m \\ &= \left[f^{n+1}\left(\tfrac{{}^nC_m}{n+m+1} f^m - \tfrac{{}^nC_{m-1}}{n+m} f^{m-1} + \cdots + \tfrac{{}^nC_0}{n+1}(-1)^m \right)\right]', \end{aligned}$$

therefore it is natural to consider the uniqueness of meromorphic functions concerning more general kind differential polnomial, such as $[f^n L(f)]^{(m)}$, where $L(z) = \lambda_l z^l + \lambda_{l-1} z^{l-1} + \cdots + \lambda_0$ and $\lambda_l \neq 0, \lambda_{l-1}, \lambda_{l-2}, \ldots, \lambda_1, \lambda_0 \neq 0$ are complex constants.

In this paper we prove the following result

Theorem 1.1. *Let f and g be two transcendental meromorphic functions such that f and g shair ∞ IM, let n, m, l be three positive integers such that $n > l + 3m + 7$. Let $L(z) = \lambda_l z^l + \lambda_{l-1} z^{l-1} + \cdots + \lambda_0$, where $\lambda_l \neq 0, \lambda_{l-1}, \lambda_{l-2}, \ldots, \lambda_1, \lambda_0 \neq 0$ are complex constants. If $[f^n L(f)]^{(m)}$ and $[g^n L(g)]^{(m)}$ share z CM then one of the following cases holds:*

(i) *$f = e^{\alpha_1}$, and $g = e^{\alpha_2}$ where α_1 and α_2 are nonconstant entire functions.*

(ii) *$f = kg$ for a constant k such that $k^p = 1$, where $p = n+l-i$, $\lambda_{l-i} \neq 0$ for some $i = 0, 1, \ldots, l$.*

(iii) *f and g satisfy algebraic euation $Q(x_1, x_2) = 0$, where*

$$Q(x_1, x_2) = x_1^n(\lambda_l x_1^l + \lambda_{l-1} x_1^{l-1} + \cdots + \lambda_0) - x_2^n(\lambda_l x_2^l + \lambda_{l-1} x_2^{l-1} + \cdots + \lambda_0)$$

.

2. Lemmas

In this section we present some lemmas which are required in the sequel.

Lemma 2.1 (see [7]). *Let f be a nonconstant meromorphic function and let $\lambda_l \neq 0, \lambda_{l-1}, \lambda_{l-2}, \ldots, \lambda_1, \lambda_0$ be small functions with respect to f. Then*

$$T(r, \lambda_l f^l + \lambda_{l-1} f^{l-1} + \cdots + \lambda_0) = lT(r, f) + S(r, f)$$

.

Lemma 2.2. *Let f, g be two nonconstant meromorphic functions sharing 1 CM and ∞ IM. Then one of the following cases holds:*

(i) *$f \equiv g$.*

(ii)

$$T(r,f) \leq N_2(r,0;f) + N_2(r,0;g) + 3\overline{N}(r,\infty;f) + S(r,f) + S(r,g)$$

and

$$T(r,g) \leq N_2(r,0;f) + N_2(r,0;g) + 3\overline{N}(r,\infty;g) + S(r,f) + S(r,g)$$

.

(iii) $fg \equiv 1$.

Proof. We omit the proof since it can be proved easily using Lemma 2.5 [1] □

Lemma 2.3 (see [6]). *Let f be a nonconstant meromorphic function and k be a positive integer. Also let c be a nonzero finite complex number. Then*

$$\begin{aligned} T(r,f) \ \leq\ & N_{k+1}(r,0;f) + \overline{N}(r,0;f^{(k)}-c) \\ & + \overline{N}(r,\infty;f) - N_0(r,0;f^{(k+1)}) + S(r,f), \end{aligned}$$

where $N_0(r,0;f^{(k+1)})$ denotes the counting function of the zeros of $f^{(k+1)}$ which are not zeros of $f(f^{(k)}-c)$.

Lemma 2.4 (see [4]). *Let f be a nonconstant meromorphic function and p,k be two positive integers. Then*

$$N_p(r,0;f^{(k)}) \leq T(r,f^{(k)}) - T(r,f) + N_{p+k}(r,0;f) + S(r,f),$$

and

$$N_p(r,0;f^{(k)}) \leq N_{p+k}(r,0;f) + k\overline{N}(r,\infty;f) + S(r,f).$$

Lemma 2.5. *Let f and g be two nonconstant meromorphic functions. Also let $F = [f^n L(f)]^{(m)}$ and $G = [g^n L(g)]^{(m)}$, where $L(z) = \lambda_l z^l + \lambda_{l-1} z^{l-1} + \cdots + \lambda_0$ and $\lambda_l \neq 0, \lambda_{l-1}, \lambda_{l-2}, \ldots, \lambda_1, \lambda_0 \neq 0$ are complex constants. If there exists three nonzero constants $\alpha_1, \alpha_2, \alpha_3$, such that $\alpha_1 F + \alpha_2 G = \alpha_3$ then $n \leq 3m + l + 3$.*

Proof. By Lemma 2.1, Lemma 2.3 and Lemma 2.4 we have

$$\begin{aligned} (n+l)T(r,f) \ \leq\ & \overline{N}(r,\infty;f) + N_{m+1}(r,0;f^n L(f)) \\ & + \overline{N}(r,0;F - \tfrac{\alpha_3}{\alpha_1}) + S(r,f) \\ \leq\ & \overline{N}(r,\infty : f) + N_{m+1}(r,0;f^n L(f)) + \overline{N}(r,0;G) + S(r,f) \\ \leq\ & \overline{N}(r,\infty : f) + N_{m+1}(r,0;f^n L(f)) + N_{m+1}(r,0;g^n L(g)) \\ & + m\overline{N}(r,\infty;g^n L(g)) + S(r,f) + S(r,g) \qquad (2.1) \\ \leq\ & \overline{N}(r,\infty : f) + N_{m+1}(r,0;f^n L(f)) \\ & + N_{m+1}(r,0;g^n L(g)) + m\overline{N}(r,\infty;g) + S(r,f) + S(r,g) \\ \leq\ & (m+l+2)T(r,f) + (2m+l+1)T(r,g) \\ & + S(r,f) + S(r,g) \end{aligned}$$

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Similarly we have

$$(n+l)T(r,g) \leq (m+l+2)T(r,g)+(2m+l+1)T(r,f) \\ +S(r,f)+S(r,g) \tag{2.2}$$

From (2.1) and (2.2) we have

$$(n-3m-3-l)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g) \tag{2.3}$$

From (2.3) we get $n \leq 3m+l+3$. □

3. Proofs of the Main Results

Proof of Theorem 1.1. Let $F = f^n L(f)$, $G = g^n L(g)$, $F_1 = [f^n L(f)]^{(m)}$, $G_1 = [g^n L(g)]^{(m)}$, $F^* = \frac{F}{z}$ and $G^* = \frac{G}{z}$. Clearly F^* and G^* share 1 CM and ∞ IM. Hence by Lemma 2.2 one of the following holds:

(i) $F^* \equiv G^*$.

(ii) $T(r,F^*) \leq N_2(r,0;F^*)+N_2(r,0;G^*)+3\overline{N}(r,\infty;F^*)+S(r,F^*)+S(r,G^*)$ and $T(r,G^*) \leq N_2(r,0;F^*)+N_2(r,0;G^*)+3\overline{N}(r,\infty;G^*)+S(r,F^*)+S(r,G^*)$.

(iii) $F^*G^* \equiv 1$.

So we have to consider the following cases.

Case I: $F^* \equiv G^*$ i.e. $F_1 \equiv G_1$. Integrating we have

$$(f^n L(f))^{(m-1)} \equiv (g^n L(g))^{(m-1)} + c_{m-1},$$

where c_{m-1} is a constant. If $c_{m-1} \neq 0$ then by Lemma 2.5 we arrive at a contradiction. Hence $c_{m-1} = 0$. Repeating the same process for $m-1$ times, we get

$$f^n L(f) \equiv g^n L(g) \tag{3.1}$$

From (3.1) we have

$$f^n(\lambda_l f^l + \lambda_{l-1} f^{l-1} + \cdots + \lambda_0) = g^n(\lambda_l g^l + \lambda_{l-1} g^{l-1} + \cdots + \lambda_0) \tag{3.2}$$

Let $k = \frac{f}{g}$.

If k is a constant then substituting $f = kg$ into (3.2) we get

$$\lambda_l g^{n+l}(k^{n+l}-1) + \lambda_{l-1} g^{n+l-1}(k^{n+l-1}-1) + \cdots + \lambda_0 g^n(k^n-1) = 0 \tag{3.3}$$

which implies that $k^p = 1$, where $p = n+l-i$, $\lambda_{l-i} \neq 0$ for some $i = 0,1,2,\ldots l$. Hence $f \equiv kg$ for a constant k, such that $k^p = 1$, where $p = n+l-i$, $\lambda_{l-i} \neq 0$ for some $i = 0,1,2,\ldots l$.

If k is not a constant, then by (3.3) f and g satisfy the algebraic equation $Q(x_1,x_2) = 0$, where

$$Q(x_1,x_2) = x_1^n(\lambda_l x_1^l + \lambda_{l-1} x_1^{l-1} + \cdots + \lambda_0) - x_2^n(\lambda_l x_2^l + \lambda_{l-1} x_2^{l-1} + \cdots + \lambda_0)$$

.

Case II: Since F^* and G^* share 1 CM and ∞ IM therefore by Lemma 2.2 we have

$$T(r,F^*) \leq N_2(r,0;F^*)+N_2(r,0;G^*)+3\overline{N}(r,\infty;F^*) \\ +S(r,F^*)+S(r,G^*) \tag{3.4}$$

and

$$T(r, G^*) \leq N_2(r, 0; F^*) + N_2(r, 0; G^*) + 3\overline{N}(r, \infty; G^*) + S(r, F^*) + S(r, G^*) \tag{3.5}$$

Without loss of generality, we suppose that $T(r, f) \leq T(r, g)$, $r \in I$, where I is a set of finite measure. By Lemma 2.1 and Lemma 2.4 we get

$$N_2(r, 0; F_1) \leq T(r, F_1) - (n + l)T(r, f) + N_{2+m}(r, 0; F) + S(r, F)$$

That is

$$N_2(r, 0; F_1) \leq T(r, F_1) - T(r, F) + N_{2+m}(r, 0; f^n L(f)) + S(r, f) \tag{3.6}$$

Since f and g are transcendental using Lemma 2.1 we have from (3.4)

$$T(r, F_1) \leq N_2(r, 0; F_1) + N_2(r, 0; G_1) + 3\overline{N}(r, \infty; F_1) + S(r, F_1) + S(r, G_1) \tag{3.7}$$

Using Lemma 2.4 from (3.6) and (3.7) we have

$$\begin{aligned}(n + l)T(r, f) &\leq N_2(r, 0; G_1) + N_{2+m}(r, 0; f^n L(f)) + 3\overline{N}(r, \infty; f) + S(r, f)\\ &\leq N_{2+m}(r, 0; g^n L(g)) + N_{2+m}(r, 0; f^n L(f))\\ &\quad +(m + 3)\overline{N}(r, \infty; f) + S(r, f)\\ &\leq (3m + 2l + 7)T(r, f) + S(r, f),\end{aligned}$$

which contradicts with $n > 3m + l + 7$.

Case III: $F^* G^* \equiv 1$. That is

$$[f^n L(f)]^{(m)} [g^n L(g)]^{(m)} \equiv z^2 \tag{3.8}$$

Since f and g share ∞ IM therefore from (3.8) it follows that f and g have no pole. Suppose,if possible, that z_0 is a zero of f of order p, then z_0 must be a zero of $[f^n L(f)]^m$ of order $np - m$. Since $n > m + 2$ therefore z_0 must be a zero of z^2 with the order at least 3. This is impossible. Therefore f has no zero. Hence $f = e^{\alpha_1}$, where α_1 is a nonconstant entire function. Similarly we can prove that $g = e^{\alpha_2}$, where α_2 is a nonconstant entire function.

This proves the theorem. □

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Competing Interests

The author declares no competing interests.

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